Fox-cartesian structure for Gray-monoidal double categories

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Applied Category Theory 2023

Overview

Gray [Gra74] defined a monoidal product for 2-categories that is left adjoint to the internal hom,

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Here we adapt *Fox-cartesian* structure to the setting of *Gray-monoidal* double categories, and describe it's theory in the graphical language of *surface diagrams*.

Locally Cubical Gray Categories

Double Categories and their Morphisms

We will need some structures from the theory of double categories:

- (preunitary weak) double categories,
- (strict and doubly-lax) functors of these,
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We can also understand them in terms of a *locally cubical Gray category*.

Locally Cubical Gray Categories – n-cells

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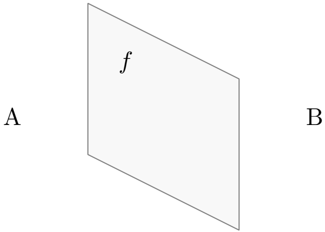
1-cells, $f : \mathrm{A} \to \mathrm{B}$,



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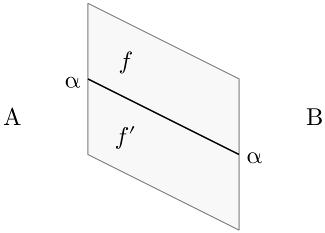

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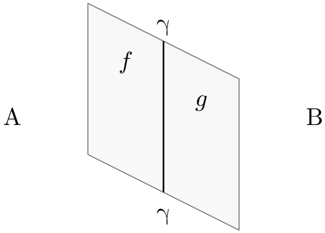

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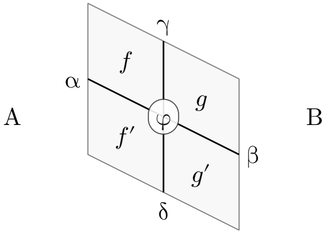
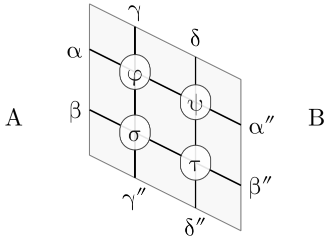
1-cells, $f, f', g, g' : A \to B$,
vertical 2-cells, $\alpha : f \to f'$, $\beta : g \to g'$,
horizontal 2-cells, $\gamma : f \nrightarrow g$, $\delta : f' \nrightarrow g'$,
3-cells, $\varphi : {}^{\gamma}_{\alpha}\Diamond^{\beta}_{\delta}$.



Locally Cubical Gray Categories – local composition

Each ordered pair of 0-cells determines a *hom double category*.



We represent their cells using string diagrams [Mye16] embedded in surfaces.

Locally Cubical Gray Categories – principal composition

For $m, n \in \mathbb{N}$ with $m + n \leq 2$,
composing an $(m+1)$-cell with 0-cell boundary $\mathrm{A} \to \mathrm{B}$
with an $(n+1)$-cell with 0-cell boundary $\mathrm{B} \to \mathrm{C}$
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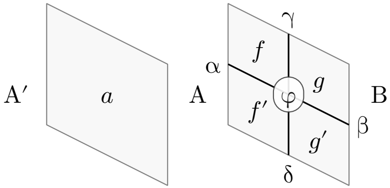
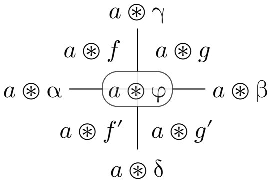
We read off the boundaries of composite cells
from the projection string diagram of a surface diagram.

Whiskerings

When $m = 0$ or $n = 0$ the principal composition is called *whiskering* $(- \circledast -)$.





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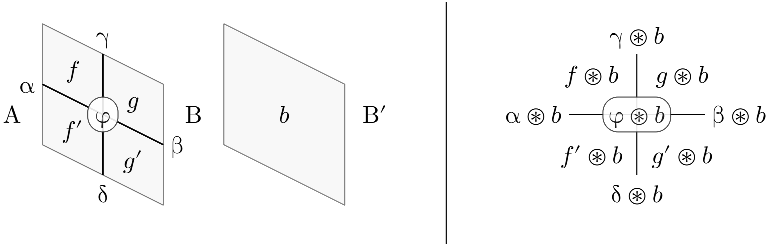

Interchangers

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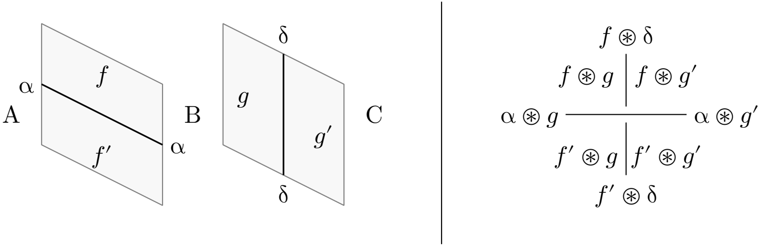

$= \quad \chi_{(\alpha,\delta)}$

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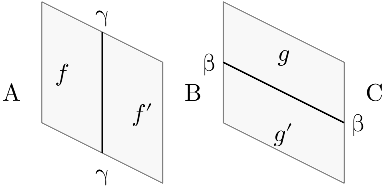

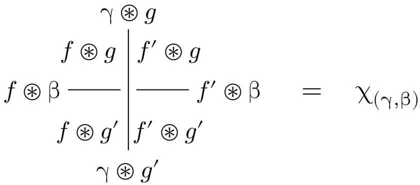

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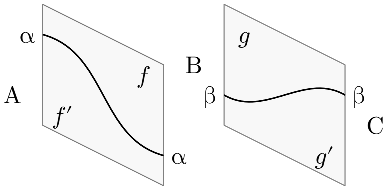

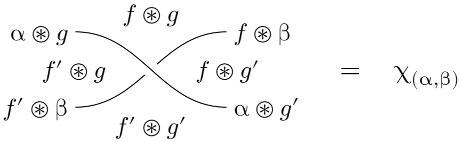

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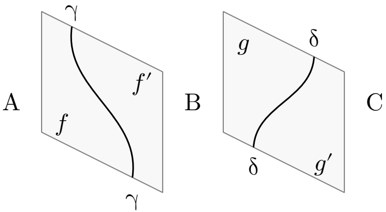

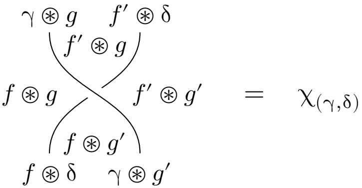

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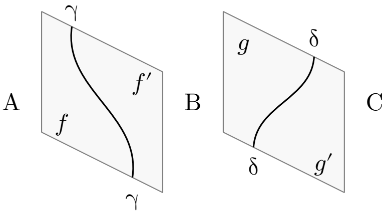
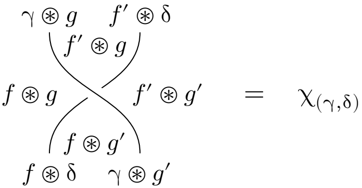




This orientation for homogeneous interchangers is called “oplax”, and its opposite “lax”.

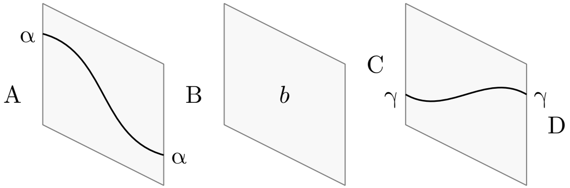
Principal Composition Laws

Principal composition is strictly unital and associative,
and compatible with local composition in hom double categories.

So long as the dimension is "in range" we can "read off" laws from surface diagrams.

E.g.

$$\chi_{(\alpha \circledast b, \gamma)} = \chi_{(\alpha, b \circledast \gamma)}$$



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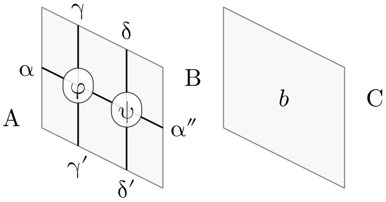

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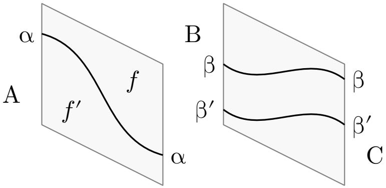
$$\chi_{(\alpha,\beta\cdot\beta')} = (\chi_{(\alpha,\beta)} \cdot \mathrm{U}(f' \circledast \beta')) \odot (\mathrm{U}(f \circledast \beta) \cdot \chi_{(\alpha,\beta')})$$



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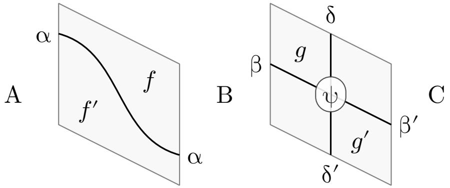
We don't get a *structure* by composing a 2-cell with a 3-cell because there are no 4-cells.



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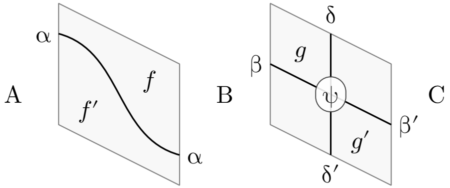
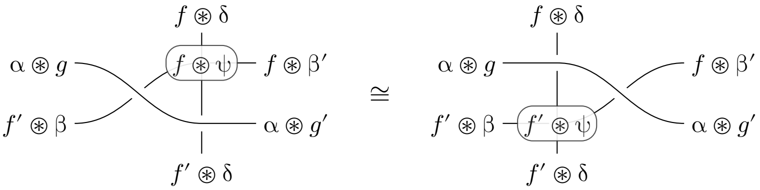


Instead we get the *property* of a naturality equation.
We can read these off of diagrams by perturbing them away from critical points [Mor22].



Locally Cubical Gray Categories of Interest

Proposition

There is a locally cubical Gray category where

0-cells are double categories,
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Proposition

A (locally globular) Gray category is a locally cubical Gray category with trivial horizontal 2-cells.

Symmetric Gray-Monoidal Double Categories

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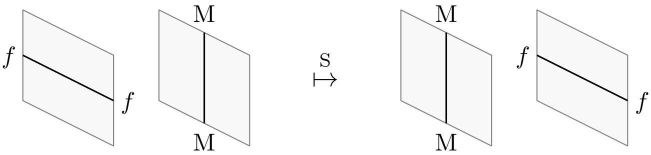
Double category $\mathbb{C}$ is Gray-monoidal if functors $\otimes_{\mathbb{C}} : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ and $\mathrm{I}_{\mathbb{C}} : \mathbb{1} \to \mathbb{C}$ form a monoid.

Braiding

The *swap functor* $\mathrm{S}_{(\mathbb{C},\mathbb{D})} : \mathbb{C} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{C}$ reverses ordered pairs.



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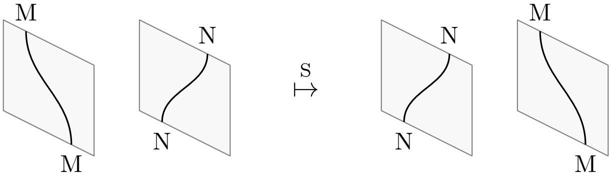

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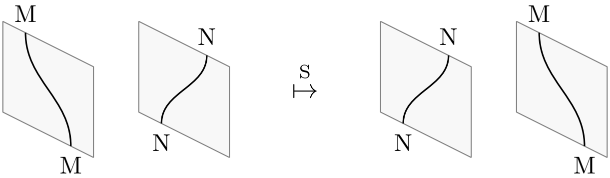
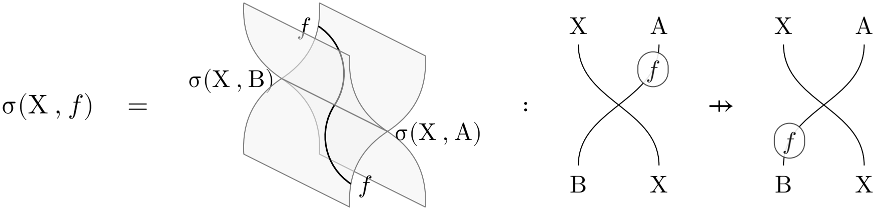


For $\mathbb{C}$ a Gray-monoidal double category with invertible interchangers, a *braiding* is a vertical pseudo transformation $\sigma : (\mathbb{C} \otimes \mathbb{C} \to \mathbb{C})\,(\otimes_{\mathbb{C}} \to S_{(\mathbb{C},\mathbb{C})} \cdot \otimes_{\mathbb{C}})$



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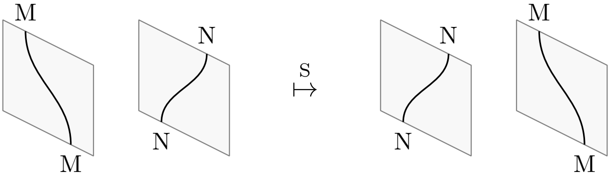

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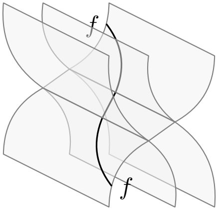

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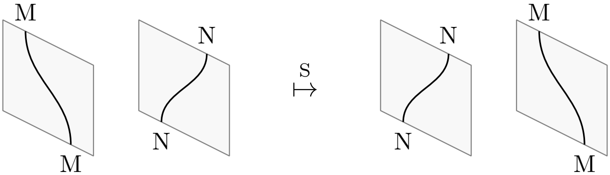

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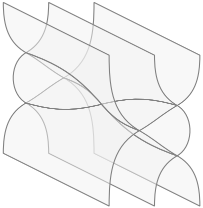
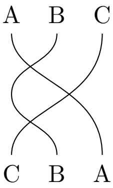
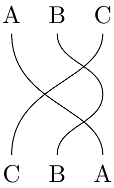
$\sigma(A\,,B\,,C) \quad =$  $:$ 

$\nrightarrow$ 

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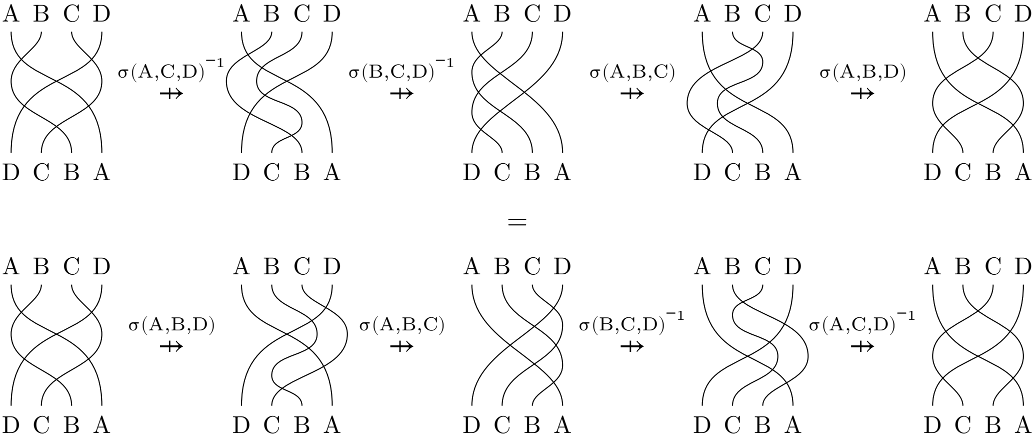
A B C D
σ(A,C,D)⁻¹
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D C B A
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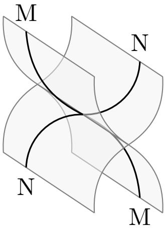
Braiding

Why do we need invertible interchangers? Consider



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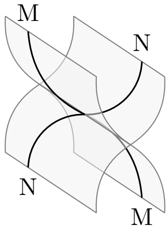
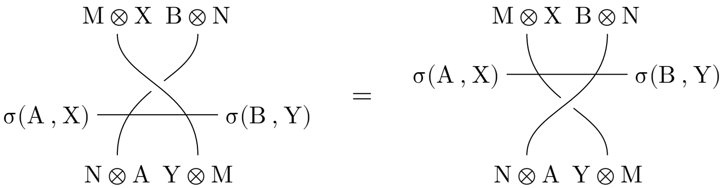
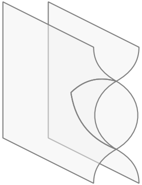
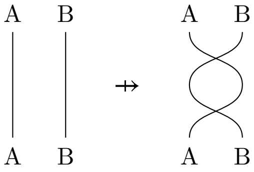


representing the relation



Syllepsis

A *syllepsis* for a braided Gray-monoidal double category $\mathbb{C}$
is an invertible globular modification $\upsilon : (\otimes_{\mathbb{C}} \to \otimes_{\mathbb{C}})(\mathrm{id}(\otimes_{\mathbb{C}}) \nrightarrow \sigma \cdot (S \cdot\cdot\, \sigma))$
relating the unbraiding to a pair of consecutive braidings

$\upsilon(A\,,B) \quad =$  $:$ 

Syllepsis

A *syllepsis* for a braided Gray-monoidal double category $\mathbb{C}$
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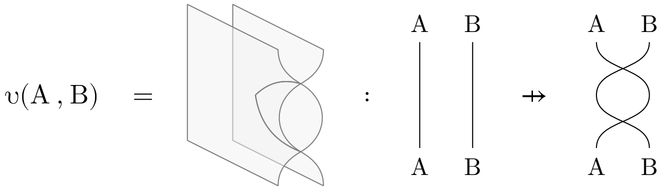
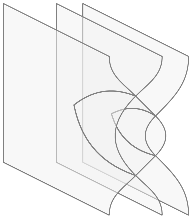


that is coherent for monoidal composition [DS97]



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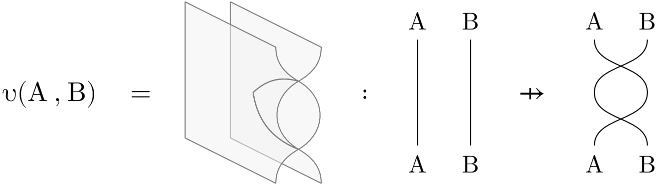

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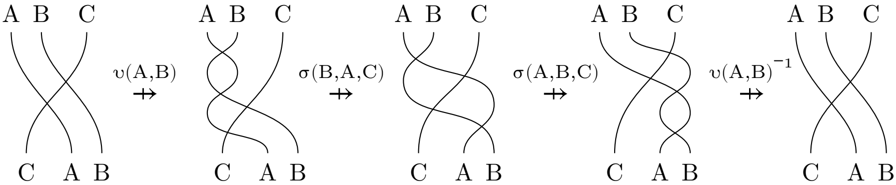

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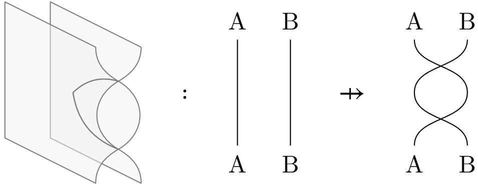
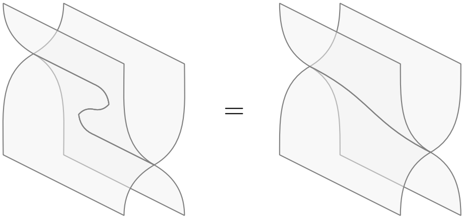


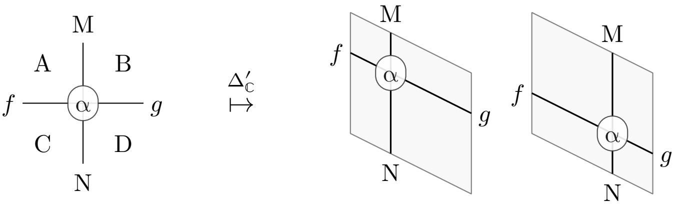
that is coherent for monoidal composition [DS97] and for Yang-Baxterators.
It is a *symmetry* if it is the unit of an adjoint equivalence.



Fox-Cartesian Structure

Gray Diagonal Functors

To permit duplication we need *Gray diagonal functors*:



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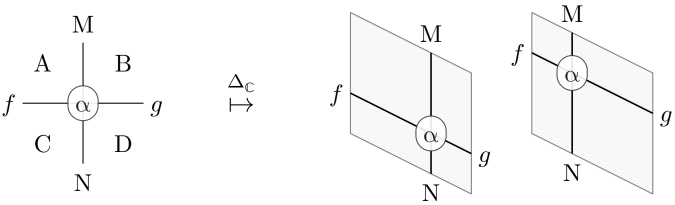

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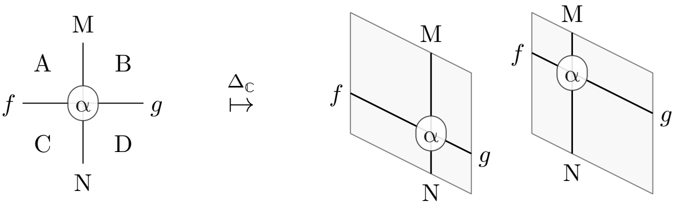
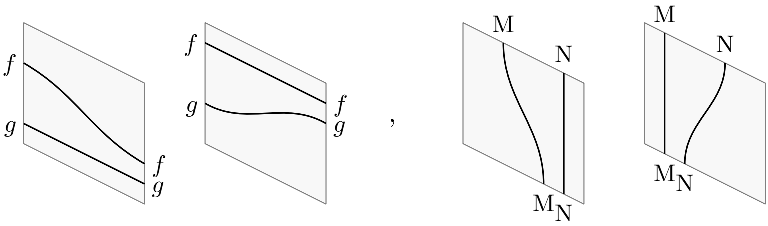
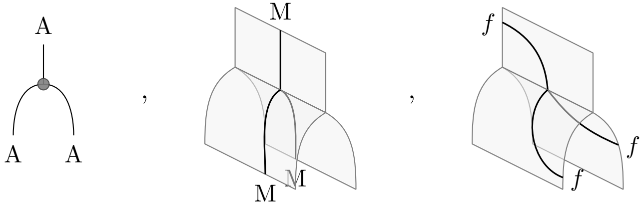


which must be *lax* in both dimensions in order to *collate* composites of 1-cells:



Duplication Stucture

From this we define *duplicator* oplax transformations of unitary pseudo functors, with components:



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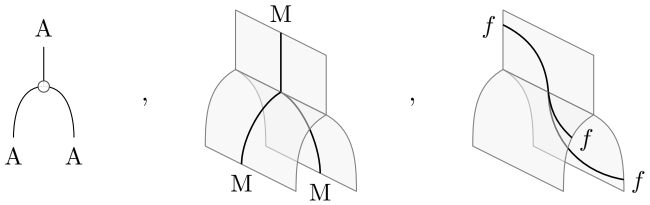

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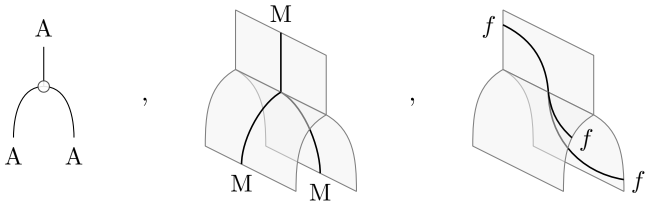
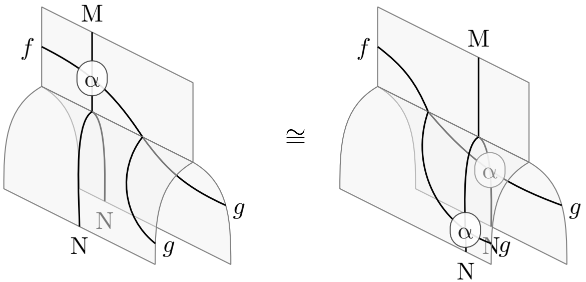


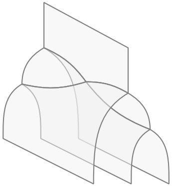
whose naturality for squares ensures:



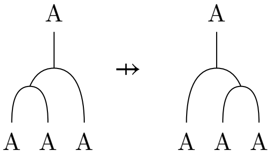
Duplication Stucture

together with globular modifications acting as *coassociator*:

$s(\mathrm{A}) \quad =$



:



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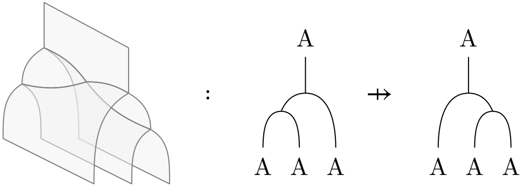
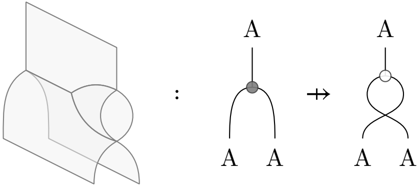


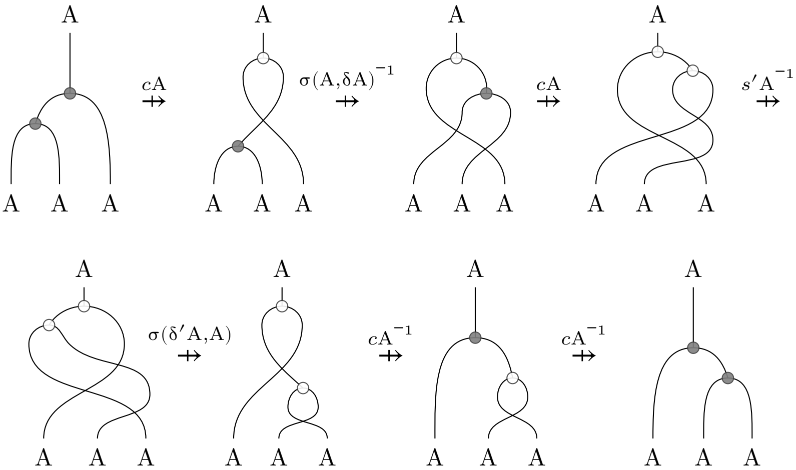
and *cocommutor*:

$c(\mathrm{A}) \quad =$



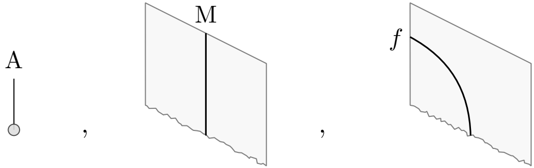
Duplication Stucture

Plus some coherences, e.g. $s(\mathrm{A}) =$



Deletion Stucture

We also define a *deletor* oplax transformation of strict functors, with components:



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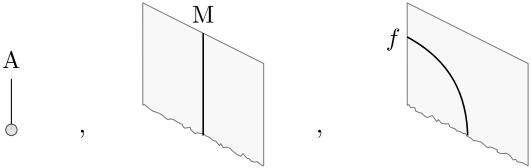

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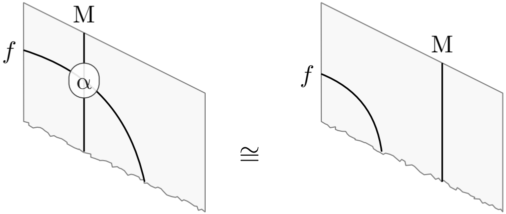
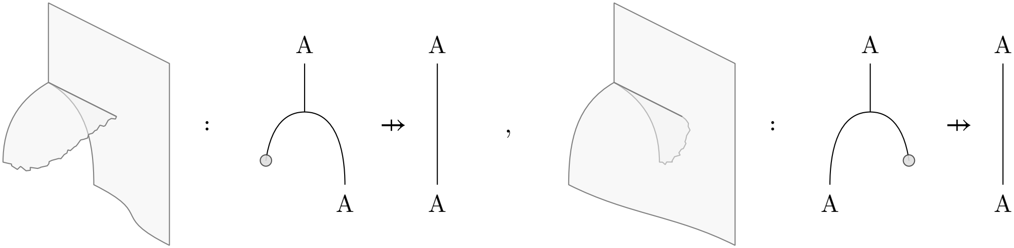


Deletion Stucture

together with modifications acting as *counitors*:



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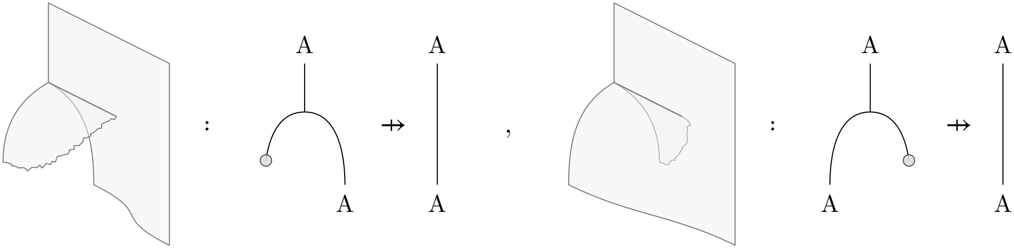

Plus coherences.

Fox-cartesian Gray-monoidal double categories

We (tentatively) propose to call symmetric Gray-monoidal double categories with duplication and deletion structure (Fox-)cartesian.

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Read more:

Cartesian Gray-Monoidal Double Categories
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Thank you!

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