Dinaturality for Double Categories

Edward Morehouse

Context

Category theory was invented to study *naturality*.[citation needed]

A first interpretation was the structure of a *natural transformation*. [EM45]

To account for “extraordinary” forms of naturality, the notion of *extranatural transformation* (née *generalized natural transformation*) was proposed. [EK66]

This was in turn generalized to *dinatural transformation*. [DS70]

Dinatural transformations have been extended to 2-*categories*. [VT10]

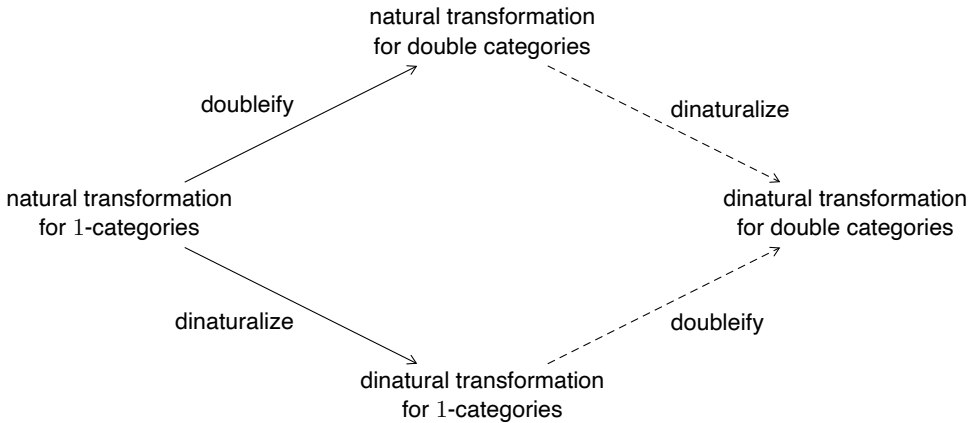
Several forms of natural transformations have been defined for double categories. [GP99]

Here we define dinatural transformations (and dimodifications) for double categories, discuss their properties and relationship to ordinary natural transformations (and modifications), and use them to characterize extranatural transformations for 1-categories. [Mor26]

Structure

Dinatural Transformations for Double Categories, Conceptually

A dinatural transformation for double categories should be a canonical common generalization or "pushout":



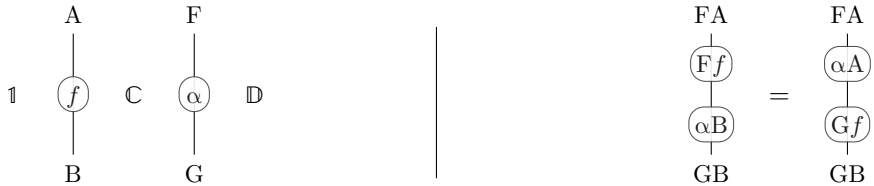
Natural Transformations for 1-Categories

A *natural transformation* of functors of 1-categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \to \mathrm{G})$ has structure:

object-component arrows for each $\mathrm{A} : \mathbb{C}$ an $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \to \mathrm{GA})$

These must satisfy the property:

naturality for arrows For $\mathbb{C}$-arrow $f : \mathrm{A} \to \mathrm{B}$,



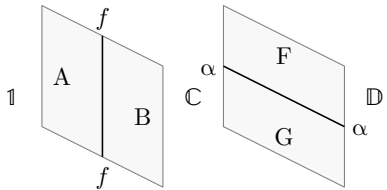
obtained by perturbing the string diagram to make the projections of α and f disjoint.

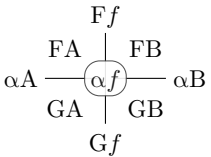
Natural Transformations for Double Categories (structure)

A (pseudo proarrow-dimension) *natural transformation* of functors of double categories $\alpha : (\mathbb{C} \to \mathbb{D})\,(\mathrm{F} \nrightarrow \mathrm{G})$ has structure:

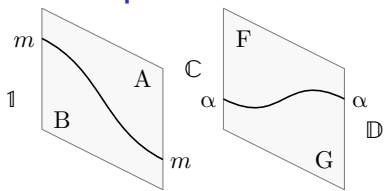
object-component proarrows for each $\mathrm{A} : \mathbb{C}$ an $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{FA} \nrightarrow \mathrm{GA})$

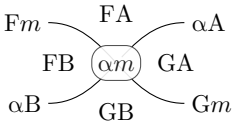
arrow-component squares for each $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ an αf





proarrow-component disks for each $m : \mathbb{C}\,(\mathrm{A} \nrightarrow \mathrm{B})$ an invertible αm

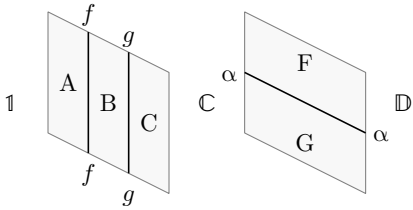


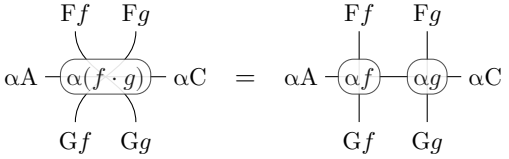


Natural Transformations for Double Categories (laws$_1$)

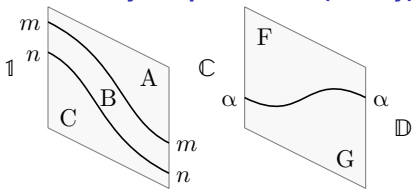
These must satisfy the properties:

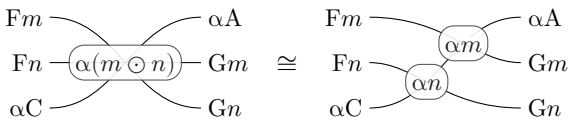
functoriality for arrows (binary) for consecutive $\mathbb{C}$-arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$,





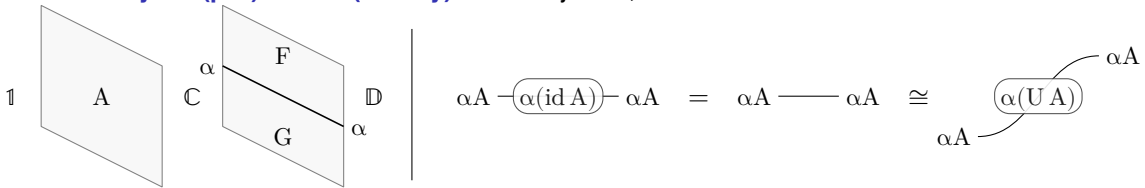
functoriality for proarrows (binary) for consecutive $\mathbb{C}$-proarrows $m : \mathrm{A} \nrightarrow \mathrm{B}$ and $n : \mathrm{B} \nrightarrow \mathrm{C}$,



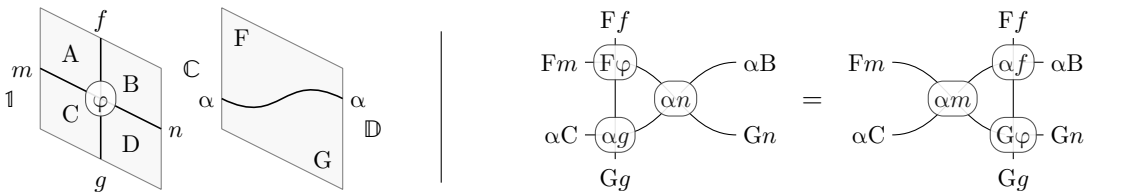


Natural Transformations for Double Categories (laws$_2$)

functoriality for (pro)arrows (nullary) for $\mathbb{C}$-object A,



naturality for squares for $\mathbb{C}$-square φ : $\left(\begin{smallmatrix}\mathrm{A}\to\mathrm{B}\\ \mathrm{A}\to\mathrm{C}\end{smallmatrix}\diamondsuit\begin{smallmatrix}\mathrm{B}\mapsto\mathrm{D}\\ \mathrm{C}\to\mathrm{D}\end{smallmatrix}\right)\left(\begin{smallmatrix}f\\ m\end{smallmatrix}\diamondsuit\begin{smallmatrix}n\\ g\end{smallmatrix}\right)$,



obtained by perturbing the surface diagram to make the projections of α and φ disjoint.

Dimensional Reflection

A *dimensional reflection* of a categorical structure swaps boundaries and reverses composition order of cells in some dimension.

E.g. each double category $\mathbb{C}$ has a *proarrow-dimension reflection* $\mathbb{C}^\bullet$ ("$\mathbb{C}$-co") with



A cartesian product with a dimensional reflection is a *dual pairing*. (e.g. $\mathbb{C}^\bullet \times \mathbb{C}$)

A functor with a dual pairing domain is a *difunctor*. (e.g. $\mathrm{F} : \mathbb{C}^\bullet \times \mathbb{C} \to \mathbb{D}$)

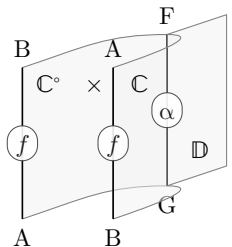
Dinatural Transformations for 1-Categories

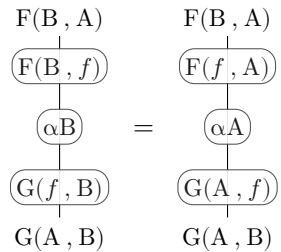
A *dinatural transformation* of difunctors of 1-categories $\alpha : (\mathbb{C}^\circ \times \mathbb{C} \to \mathbb{D})\,(\mathrm{F} \ddot{\to} \mathrm{G})$ has structure:

object-component arrows for each $\mathrm{A} : \mathbb{C}$ an $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{F}(\mathrm{A}\,,\,\mathrm{A}) \to \mathrm{G}(\mathrm{A}\,,\,\mathrm{A}))$

These must satisfy the property:

dinaturality for arrows For arrow $f : \mathrm{A} \to \mathrm{B}$ of $\mathbb{C}$,





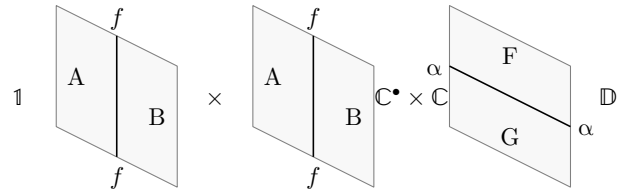
obtained by perturbing the surface diagram to make the projections of α and the fs disjoint, while observing the *diagonal restriction*: the dual pairing must agree at a dinatural element.

Dinatural Transformations for Double Categories (structure)

A (pseudo) *proarrow-dimension dinatural transformation* of difunctors of double categories $\alpha : (\mathbb{C}^\bullet \times \mathbb{C} \to \mathbb{D})\,(\mathrm{F} \ddot{\nrightarrow} \mathrm{G})$ has structure:

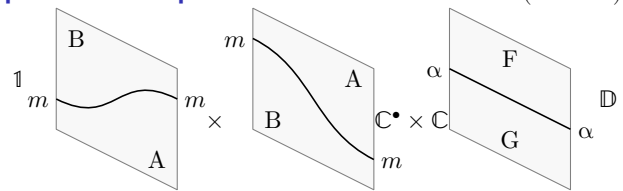
object-component proarrows for each $\mathrm{A} : \mathbb{C}$ an $\alpha\mathrm{A} : \mathbb{D}\,(\mathrm{F}(\mathrm{A}\,,\,\mathrm{A}) \nrightarrow \mathrm{G}(\mathrm{A}\,,\,\mathrm{A}))$

arrow-component squares for each $f : \mathbb{C}\,(\mathrm{A} \to \mathrm{B})$ an αf



$$\begin{array}{ccc} & \mathrm{F}(f\,,\,f) & \\ \mathrm{F}(\mathrm{A}\,,\,\mathrm{A}) & | & \mathrm{F}(\mathrm{B}\,,\,\mathrm{B}) \\ \alpha\mathrm{A} & \text{—}(\alpha f)\text{—} & \alpha\mathrm{B} \\ \mathrm{G}(\mathrm{A}\,,\,\mathrm{A}) & | & \mathrm{G}(\mathrm{B}\,,\,\mathrm{B}) \\ & \mathrm{G}(f\,,\,f) & \end{array}$$

proarrow-component disks for each $m : \mathbb{C}\,(\mathrm{A} \nrightarrow \mathrm{B})$ an invertible αm

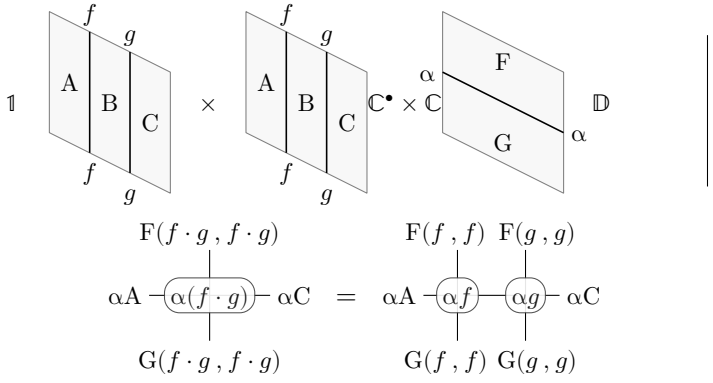


$$\begin{array}{ccccc} & & \mathrm{F}(\mathrm{B}\,,\,\mathrm{A}) & & \\ \mathrm{F}(\mathrm{B}\,,\,m) & & & & \mathrm{F}(m\,,\,\mathrm{A}) \\ & \mathrm{F}(\mathrm{B}\,,\,\mathrm{B}) & & \mathrm{F}(\mathrm{A}\,,\,\mathrm{A}) & \\ & \alpha\mathrm{B} & \text{—}(\alpha m)\text{—} & \alpha\mathrm{A} & \\ & \mathrm{G}(\mathrm{B}\,,\,\mathrm{B}) & & \mathrm{G}(\mathrm{A}\,,\,\mathrm{A}) & \\ \mathrm{G}(m\,,\,\mathrm{B}) & & & & \mathrm{G}(\mathrm{A}\,,\,m) \\ & & \mathrm{G}(\mathrm{A}\,,\,\mathrm{B}) & & \end{array}$$

Dinatural Transformations for Double Categories (laws$_1$)

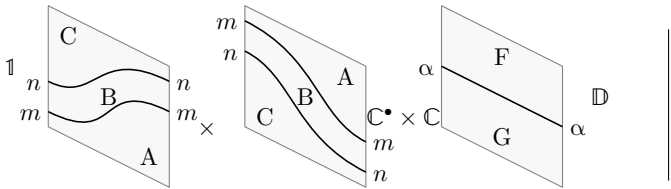
These must satisfy the properties:

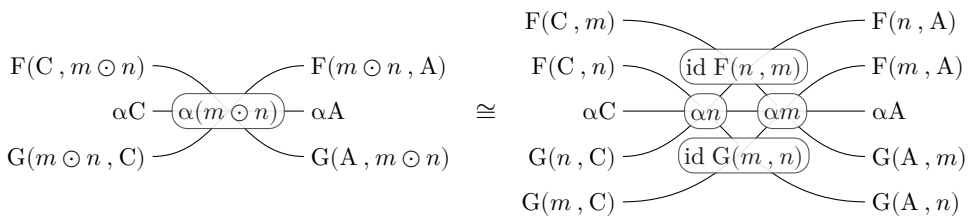
functoriality for arrows (binary) for consecutive $\mathbb{C}$-arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$,



Dinatural Transformations for Double Categories (laws$_2$)

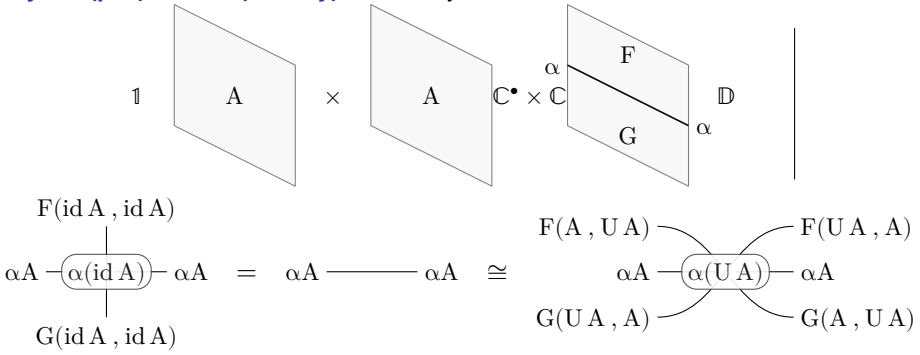
functoriality for proarrows (binary) for consecutive $\mathbb{C}$-proarrows $m : \mathrm{A} \nrightarrow \mathrm{B}$ and $n : \mathrm{B} \nrightarrow \mathrm{C}$,





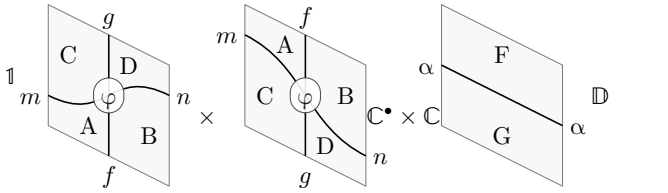
Dinatural Transformations for Double Categories (laws$_3$)

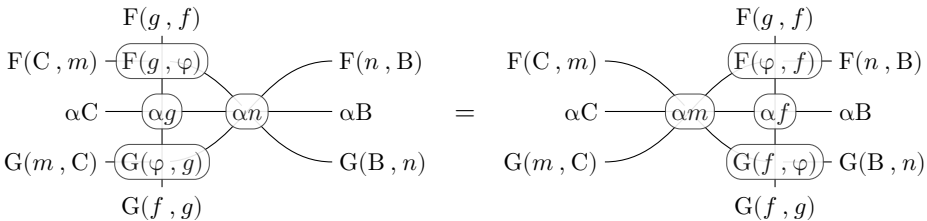
functoriality for (pro)arrows (nullary) for $\mathbb{C}$-object A,



Dinatural Transformations for Double Categories (laws_4)

naturality for squares for $\mathbb{C}$-square φ : $\left(\begin{smallmatrix} A \to B \\ A \nrightarrow C \end{smallmatrix} \diamondsuit \begin{smallmatrix} B \nrightarrow D \\ C \to D \end{smallmatrix}\right) \left(\begin{smallmatrix} f \\ m \end{smallmatrix} \diamondsuit \begin{smallmatrix} n \\ g \end{smallmatrix}\right)$,





obtained by perturbing the surface diagram to make the projections of α and the φs disjoint, while observing the *diagonal restriction*: the dual pairing must agree at a dinatural element.

Degeneration Sanity Checks

It is easy to check that a dinatural transformation for degenerate double categories

- where all squares are globular is a dinatural transformation for 2-categories[1],
- where all such disks are identities is a dinatural transformation for 1-categories.

[1] modulo details

Properties

Dinatural Transformation Properties

Dinatural transformations for double categories enjoy many of the same properties as dinatural transformations for 1-categories, including:

- They can always be composed with natural transformations.
- They can be composed with each other provided that the 0-dimensional domain is groupoidal in the diagonalized dimension.
- There are boundary-respecting bijections between the families of sets:
  ordinary (functors / natural transformations / modifications) and
  dummy (difunctors / dinatural transformations / dimodifications),
  where a cell with 0-dimensional boundary $\mathbb{B} \times \mathbb{C} \to \mathbb{D}$ is *dummy* in one index if it factors through the projection to the other.

So dinaturality subsumes ordinary naturality as its dummy fragment for double categories, just as for 1-categories.

Slogan: naturality is just dummy dinaturality.

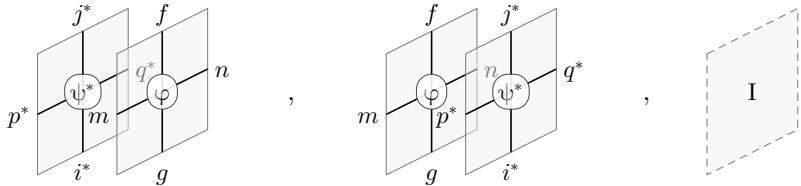
Applications

Difunctors of Interest

For a monoidal double category $\mathbb{C}$ with a functor $^* : \mathbb{C}^\bullet \to \mathbb{C}$,

define difunctors $^* \otimes\ ,\ \otimes\ ^*, \mathrm{I} : \mathbb{C}^\bullet \times \mathbb{C} \to \mathbb{C}$

sending $(\psi\ ,\ \varphi)$ respectively to $\psi^* \otimes \varphi$, $\varphi \otimes \psi^*$, and I:



Folds

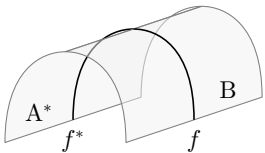
We call dinatural transformations

$$\eta : (\mathbb{C}^{\bullet} \times \mathbb{C} \to \mathbb{C})\,(\mathrm{I} \ddot{\nrightarrow} {}^{*} \otimes) \quad \text{and} \quad \varepsilon : (\mathbb{C}^{\bullet} \times \mathbb{C} \to \mathbb{C})\,(\otimes^{*} \ddot{\nrightarrow} \mathrm{I})$$

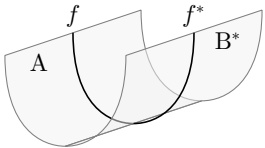
respectively a *cap* and *cup fold* for $(\mathbb{C}, {}^{*})$. They have:

object-component proarrows: $\eta \mathrm{A} : \mathrm{I} \nrightarrow \mathrm{A}^{*} \otimes \mathrm{A}$ and $\varepsilon \mathrm{A} : \mathrm{A} \otimes \mathrm{A}^{*} \nrightarrow \mathrm{I}$

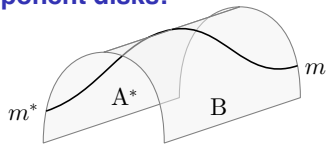
arrow-component squares:



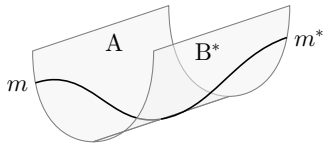
,



proarrow-component disks:

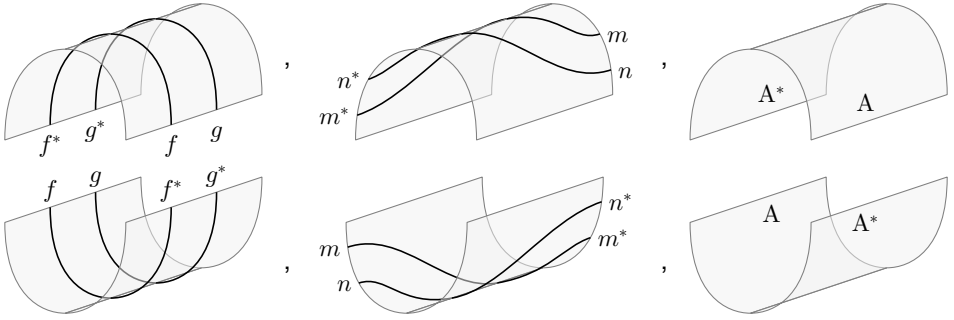


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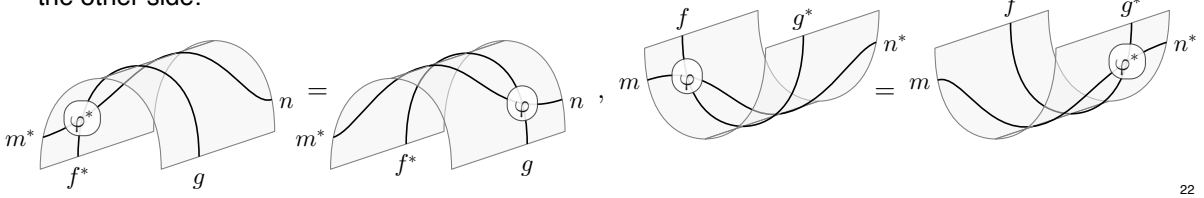


Fold Laws

(Pro)arrow functoriality gives unique interpretations for multiple 1-cells wrapped around a fold:



Square naturality says that a square with half its boundary wrapped around a fold can be flipped to the other side:



Strips

We always have the identity natural transformations

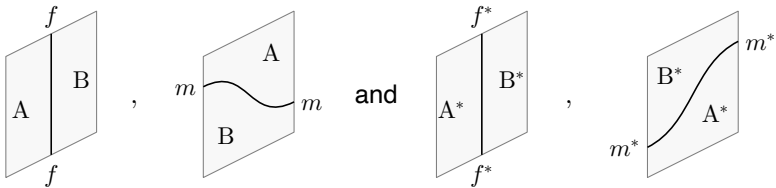
$$\mathrm{U}\,\mathrm{id} : (\mathbb{C} \to \mathbb{C})\,(\mathrm{id} \nrightarrow \mathrm{id}) \quad \text{and} \quad \mathrm{U}^* : (\mathbb{C}^\bullet \to \mathbb{C})\,(* \nrightarrow *)$$

These are the same thing as dummy dinatural transformations

$$\iota : (\mathbb{C}^\bullet \times \mathbb{C} \to \mathbb{C})\,(\pi_1 \overset{\cdot\cdot}{\nrightarrow} \pi_1) \quad \text{and} \quad \iota^* : (\mathbb{C}^\bullet \times \mathbb{C} \to \mathbb{C})\,(\pi_0 \cdot * \overset{\cdot\cdot}{\nrightarrow} \pi_0 \cdot *)$$

that we call *strips*.

Their 1-cell components are all identity disks.



Folds in CAT

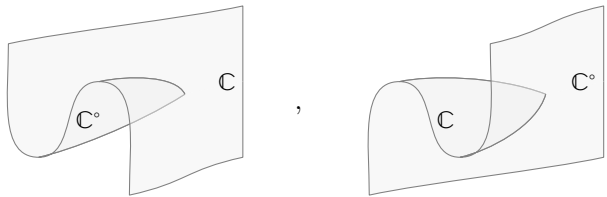
The cartesian monoidal double category of (small) 1-categories, CAT, with the oppositization functor, $^\circ : \mathrm{CAT}^\bullet \to \mathrm{CAT}$, has cap and cup folds,

$$\eta : (\mathrm{CAT}^\bullet \times \mathrm{CAT} \to \mathrm{CAT})\,(\mathbb{1} \;\ddot{\nrightarrow}\; {}^\circ \times\,) \quad , \quad \varepsilon : (\mathrm{CAT}^\bullet \times \mathrm{CAT} \to \mathrm{CAT})\,(\,\times {}^\circ \;\ddot{\nrightarrow}\; \mathbb{1})$$

with category-component profunctors,

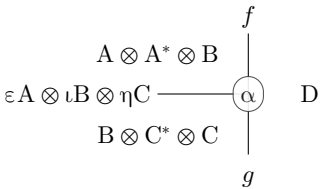
$$\eta\mathbb{C} : \mathbb{1} \nrightarrow \mathbb{C}^\circ \times \mathbb{C} \;:\cong\; \mathbb{C}\,(- \to -) : \mathbb{C}^\circ \times \mathbb{C} \to \mathrm{SET} \;\cong:\; \varepsilon\mathbb{C} : \mathbb{C} \times \mathbb{C}^\circ \nrightarrow \mathbb{1}$$

These form the unit and counit of a categorified adjunction:

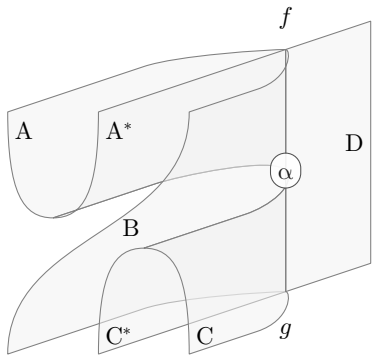


Extranatural Squares and Transformations

We call a square α *extranatural* if it has boundary



or



An extranatural square in CAT is an *extranatural transformation*, where the extranaturality laws follow from square naturality for the dinatural transformations ε, ι, and η.

References

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