Double-Categorical Models of Directed Universes

Ed Morehouse
(joint work with Alex Kavvos and Dan Licata)

TallCat Seminar

2019.12.12

Identity Types

In type theory, *identity types* classify paths between elements of a type.

$$\frac{\Gamma \vdash \mathrm{A} : \mathcal{U} \qquad \Gamma \vdash x : \mathrm{A} \qquad \Gamma \vdash y : \mathrm{A}}{\Gamma \vdash x =_{\mathrm{A}} y : \mathcal{U}} = \downarrow$$

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Thus elements of an identity type form undirected (bidirected) paths.

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Paths between types have *computational content* because we can *transport* structure along a them:

$$\frac{\Gamma \vdash p : \mathrm{A} =_{\mathcal{U}} \mathrm{B} \qquad \Gamma \vdash x : \mathrm{A}}{\Gamma \vdash \mathrm{tr}(p)(x) : \mathrm{B}}$$

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A path type in a universe relates the types at its two distinct endpoints.

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where pathy is a mere predicate on functions.

As in HoTT, each path has an underlying function.

We *transport* along a path by applying that function.

Slogan: “all paths are transportable”.

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We do this using double categories with connection structure.

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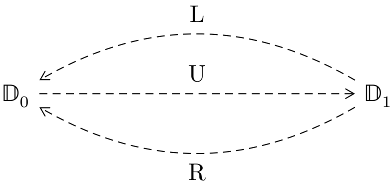

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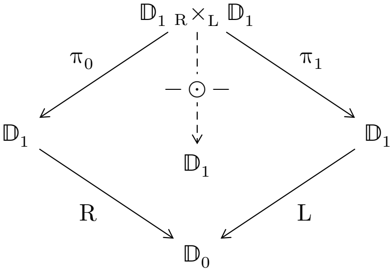

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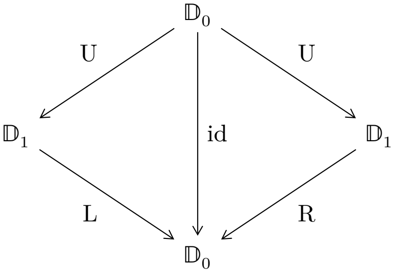

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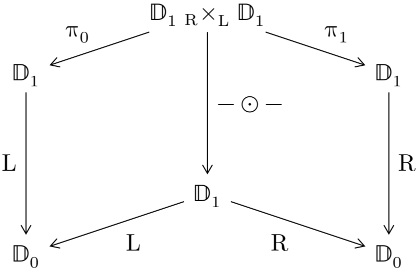

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and come equipped with unitor and associator natural isomorphisms

$$\mathrm{U(LM)} \odot \mathrm{M} \cong \mathrm{M} \cong \mathrm{M} \odot \mathrm{U(RM)} \qquad (\mathrm{M} \odot \mathrm{N}) \odot \mathrm{P} \cong \mathrm{M} \odot (\mathrm{N} \odot \mathrm{P})$$

satisfying the triangle and pentagon equations

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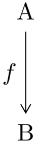
1-dimensional *arrow* (vertical) morphisms:



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1-dimensional *proarrow* (horizontal) morphisms:

$$A \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} C$$

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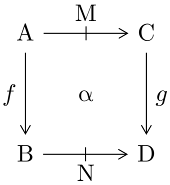
2-dimensional squares:



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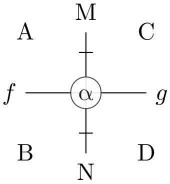
2-dimensional squares – which we can draw using dual diagrams [Mye07]:



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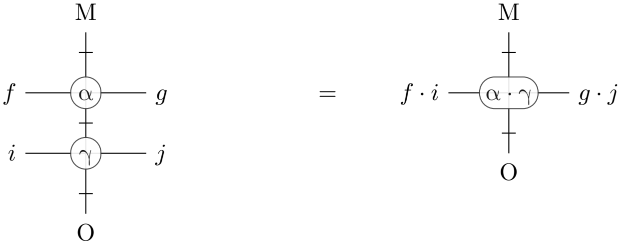
(strict) composition in the arrow dimension:



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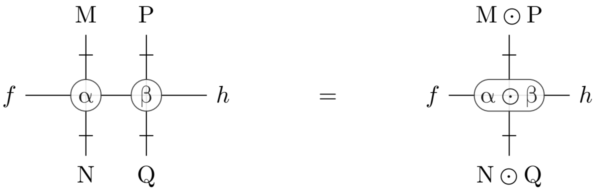

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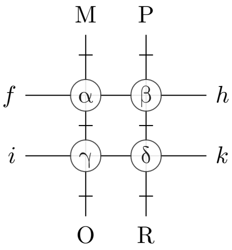
generalized associativity [DP93]:



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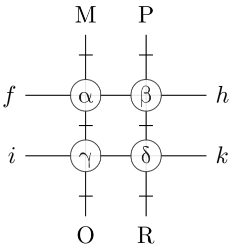

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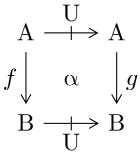
A coherence theorem allows us to ignore and recover coherators [GP99].

Globular Squares

A square with trivial boundary in some dimension is a *globe*.



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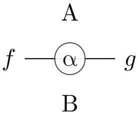

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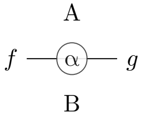

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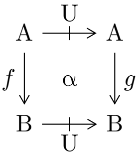

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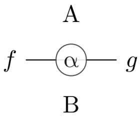

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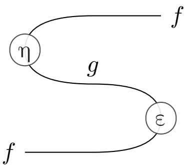
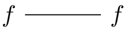
An *arrow adjunction* in a double category is formed by arrows $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{A}$ and arrow globes $\eta : \mathrm{id}(\mathrm{A}) \nrightarrow f \cdot g$ and $\varepsilon : g \cdot f \nrightarrow \mathrm{id}(\mathrm{B})$ such that:



$\cong$



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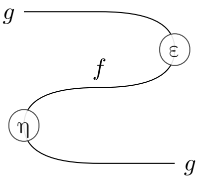

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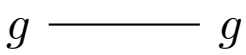

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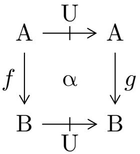

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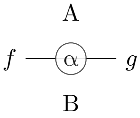

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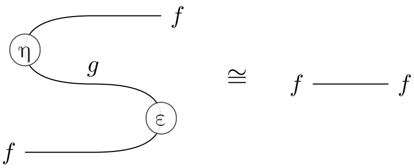

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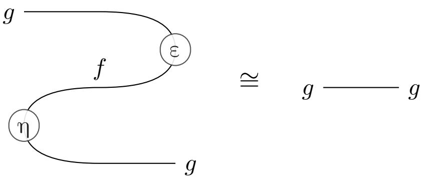

(The "$- \cong -$" accounts for boundary coherators.)

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For proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ and object C there is a natural bijection between squares from M to the identity proarrow on C and arrows from the cotabulator of M to C:

$$\frac{\alpha : \mathbb{D}_1\,(\mathrm{M} \to \mathrm{U}(\mathrm{C}))}{d : \mathbb{D}_0\,(\perp(\mathrm{M}) \to \mathrm{C})}$$

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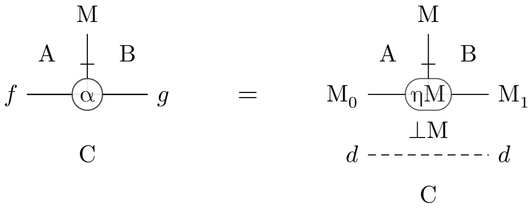
The adjunction unit component at M is a morphism $\eta\mathrm{M} : \mathbb{D}_1\,(\mathrm{M} \to \mathrm{U}(\perp\mathrm{M}))$ with the universal property:



Connection Structure

In a double category, parallel arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ are *companions* [GP04] if there are *connection squares*

$$\begin{array}{ccc} \mathrm{A} & \overset{\mathrm{M}}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \\ f \big\downarrow & {}^{\cdot} f_{\lrcorner} & \big\downarrow \mathrm{id} \\ \mathrm{B} & \underset{\mathrm{U}}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{A} & \overset{\mathrm{U}}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & \mathrm{A} \\ \mathrm{id} \big\downarrow & {}^{\ulcorner} f_{.} & \big\downarrow f \\ \mathrm{A} & \underset{\mathrm{M}}{\longrightarrow\!\!\!\!\!\!\!+\;\;\;} & \mathrm{B} \end{array}$$

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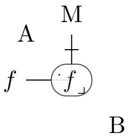

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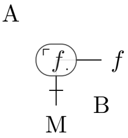

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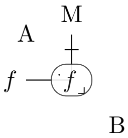

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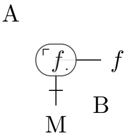
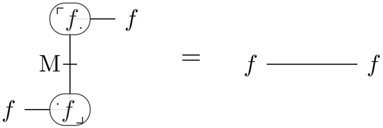


satisfying the *companion laws*:



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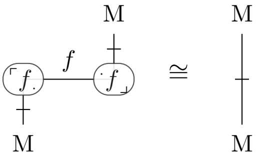

Companionship Properties

When they exist, companion morphisms are unique up to canonical isoglobes.
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Companionship respects morphism composition structure:

$$\widehat{f \cdot g} = \hat{f} \odot \hat{g} \qquad \text{and} \qquad \widehat{\mathrm{id}(\mathrm{A})} = \mathrm{U}(\mathrm{A})$$

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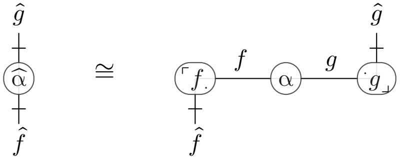

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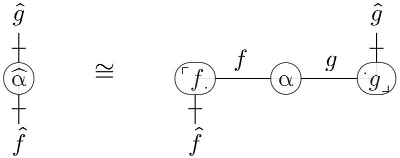

Companionship also respects globe composition structure (contravariantly).
There are $(0, 2)$-full sub-double categories of companionable arrow- and proarrow globes, which are equivalent as bicategories.

Unique Factorization by Connection Squares

For any square $\alpha : {}^{\mathrm{M}}_{f}\Diamond^{g}_{\mathrm{N}}$,

- if f is companionable then there is a unique square $\lambda : {}^{\mathrm{M}}_{\mathrm{id}}\Diamond^{g}_{\hat{f}\odot\mathrm{N}}$ with $\lambda \cdot (\llcorner f \lrcorner \odot \mathrm{U}(\mathrm{N})) \cong \alpha$,
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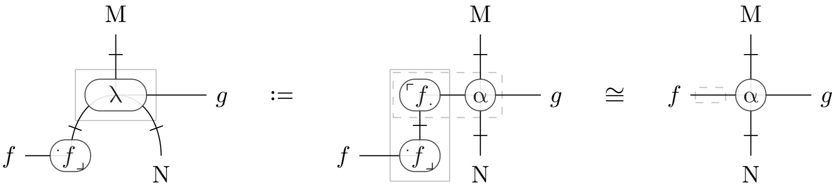

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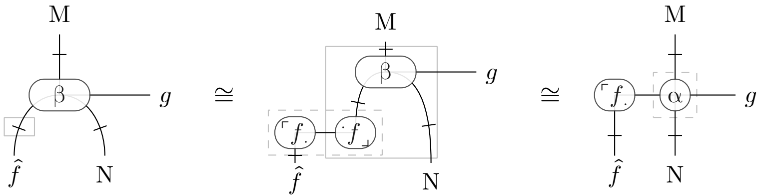
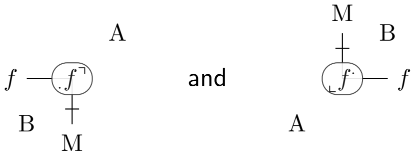


Conjoints

In a double category, antiparallel arrow $f : \mathrm{A} \to \mathrm{B}$ and proarrow $\mathrm{M} : \mathrm{B} \nrightarrow \mathrm{A}$ are *conjoints* if there are *coconnection squares*



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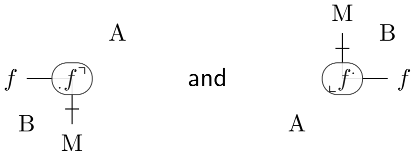

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Dual to companions, conjoints extend to globes, respect composition, and are essentially unique ($\check{f}$).

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Any two of these structures determine the third:

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If proarrow $\mathrm{M} : \mathrm{A} \nrightarrow \mathrm{B}$ has companion f and conjoint g then $f \dashv g$.

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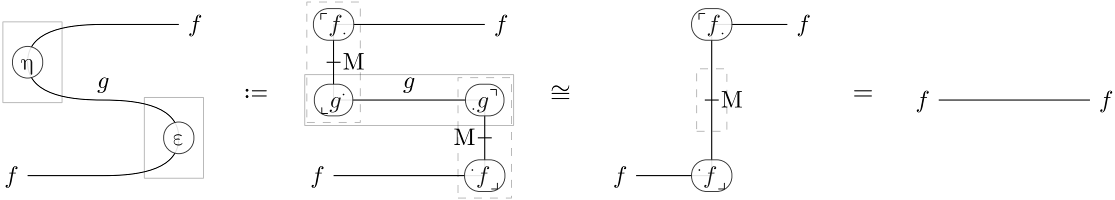

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For arrow adjunction $f \dashv g$, a proarrow M is companion to f iff it is conjoint to g.

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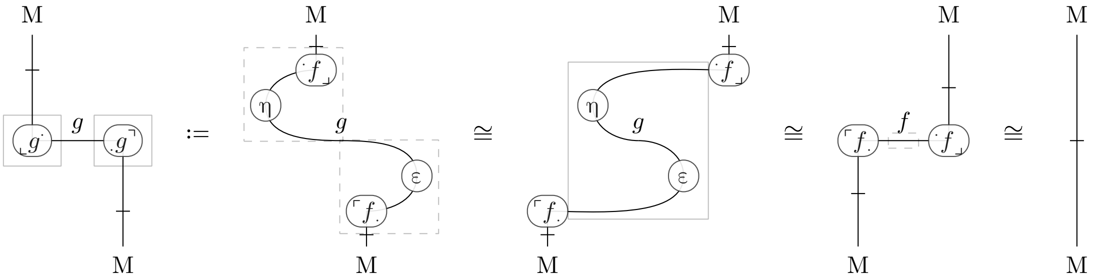

Profunctors

A *profunctor* $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ is a functor $\mathrm{M} : \mathbb{A}^\circ \times \mathbb{B} \to \textsc{Set}$.

Profunctors act as generalized relations.

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To *compose* $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ and $\mathrm{N} : \mathbb{B} \nrightarrow \mathbb{C}$ we quotient by connected components of $\mathbb{B}$ using a coend:

$$\mathrm{M} \odot \mathrm{N} : \mathbb{A} \nrightarrow \mathbb{C} \quad := \quad \int^{\mathrm{B}:\mathbb{B}} \mathrm{M}\,(\underline{\;1\;} \dashrightarrow \mathrm{B}) \times \mathrm{N}\,(\mathrm{B} \dashrightarrow \underline{\;2\;}) : \mathbb{A}^\circ \times \mathbb{C} \to \textsc{Set}$$

Composition is associative up to canonical isomorphism by the “Fubini theorem” for coends [Kel82].

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for profunctor $\mathrm{N} : \mathbb{B} \nrightarrow \mathbb{D}$ and functors $\mathrm{F} : \mathbb{A} \to \mathbb{B}$ and $\mathrm{G} : \mathbb{C} \to \mathbb{D}$,

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In particular, the hom functor $\mathbb{A}\,(\frac{1}{\ } \to \frac{2}{\ }) : \mathbb{A}^{\circ} \times \mathbb{A} \to \textsc{Set}$ is an *identity profunctor* $\mathrm{U} : \mathbb{A} \nrightarrow \mathbb{A}$.

Double Categories of Categories

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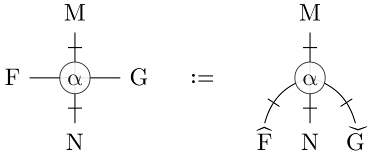

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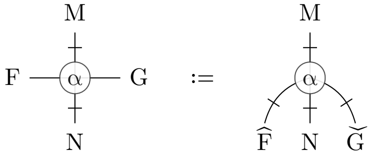

There is a double category of (small) categories, functors, profunctors, and natural squares [GP99], where companion/conjoint proarrows are covariantly/contravariantly represented profunctors [GP04].

Collage Categories

The cotabulator of a profunctor $\mathrm{M} : \mathbb{A} \nrightarrow \mathbb{B}$ is its *collage* [GP99], a category $\mathrm{Col}\,\mathrm{M}$ with:

objects the disjoint union of $\mathbb{A}$-objects and $\mathbb{B}$-objects,

arrows the disjoint union of $\mathbb{A}$-homomorphisms, $\mathbb{B}$-homomorphisms and M-heteromorphisms.

Identities and homomorphism compositions are inherited from $\mathbb{A}$ and $\mathbb{B}$.
Mixed composition follows by M functoriality: for $a : \mathbb{A}\,(\mathrm{A}' \to \mathrm{A})$, $x : \mathrm{M}\,(\mathrm{A} \dashrightarrow \mathrm{B})$, $b : \mathbb{B}\,(\mathrm{B} \to \mathrm{B}')$,

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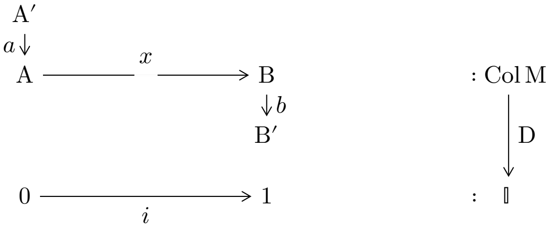

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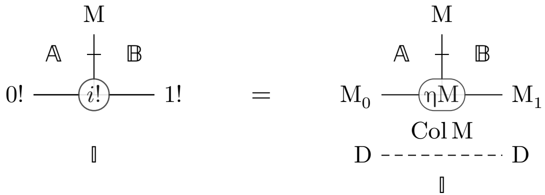

Interval Displays for Represented Profunctors

In the collage category of a covariantly represented profunctor, $\mathrm{Col}\,\widehat{\mathrm{F}}$, the morphisms $\mathrm{id}(\mathrm{FA}) : \widehat{\mathrm{F}}\,(\mathrm{A} \dashrightarrow \mathrm{FA})$ are opcartesian for the display functor $\mathrm{D} : \mathrm{Col}\,\widehat{\mathrm{F}} \to \mathbb{I}$.

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Thus the interval display functor is a Grothendieck opfibration.

This is equivalent to a pseudofunctor $\mathbb{I} \to \mathrm{C{\scriptstyle AT}}$ sending $i : \mathbb{I}\,(0 \to 1)$ to the representing functor $\mathrm{F} : \mathbb{A} \to \mathbb{B}$.

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Every proarrow is companionable, so every path is transportable.

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This gives us the following univalence-like principle for types A and B in a universe $\mathcal{U}$:

$$\overbrace{(\mathrm{A} \rightsquigarrow_{\mathcal{U}} \mathrm{B})}^{\text{directed paths in } \mathcal{U} \text{ from A to B}} \quad \simeq \quad \underbrace{(f : \mathrm{A} \to \mathrm{B} \mid \mathrm{comp}\, f)}_{\text{companionable functions from A to B}}$$

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This occurs precisely when the profunctor is contravariantly represented,

which happens when it has a covariantly represented left adjoint.

Kan Structure

Connection structure on a double category interprets a form of Kan structure for the universe.

For arbitrary path and composable companionable functions:

$$\begin{array}{ccc} A & & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow[\mathrm{N}]{\quad +\quad} & D \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{M}} & C \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g} \\ A & & D \end{array}$$

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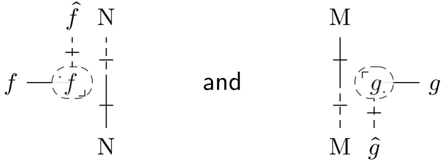
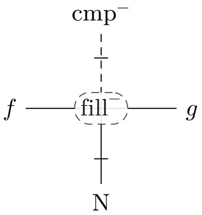


These are universal, in the sense that any other fillers factor uniquely through them.

Contravariant Paths

In order to fill the following "cubical horn" we need a path corresponding to the function g "backwards":



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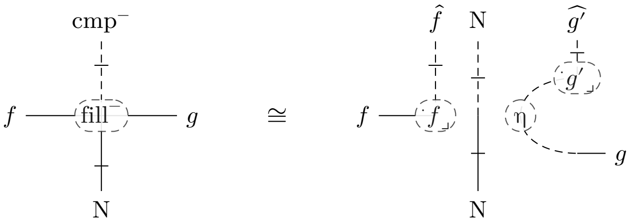

If g is conjoinable then we can fill the square.

Path Structures on Universes

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Some examples:

- if only identity arrows are companionable then the universe is path-discrete,
- if all arrows are companionable, then we have a directed path structure mirroring the function structure,
- if arrows of an adjoint equivalence are companionable, then all paths are bi-directional (because all companionable arrows then have companionable left adjoints).

References

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Bonus Slides

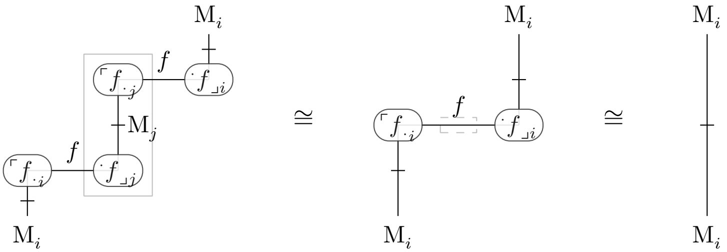
Companion Uniqueness

Theorem

If proarrows $M_0, M_1 : A \nrightarrow B$ are each companion to arrow $f : A \to B$ then $M_0 \cong M_1$.

Proof.

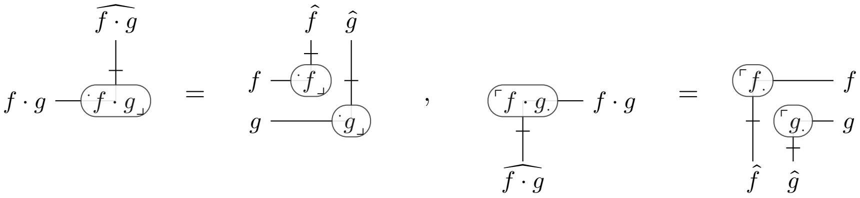
For $\{i, j\} = \{0, 1\}$, we have:



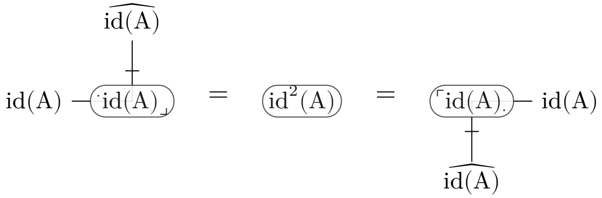
So the proarrow globes $\ulcorner f_{\cdot 1} \odot {}^{\cdot} f_{\lrcorner 0} : M_0 \to M_1$ and $\ulcorner f_{\cdot 0} \odot {}^{\cdot} f_{\lrcorner 1} : M_1 \to M_0$ form an isomorphism. □

Companion Compositionality

It is easily checked that the companion laws are satisfied with

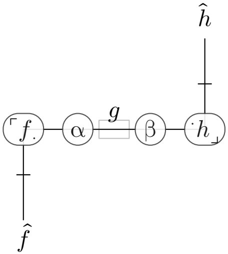


and

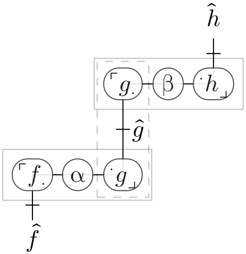


Companion Globe Compositionality

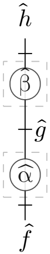
$\widehat{\alpha \odot \beta} \cong$



$\cong$



$\cong$



$= \hat{\beta} \cdot \hat{\alpha}$