Cubical Structures for Higher-Dimensional Type Theories

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- A *higher-dimensional type theory* depends on a notion of *higher-dimensional abstract spaces*.
- Many choices: globular, simplicial, cubical, opotopic, etc.
- We want abstract spaces with good topological properties as well as good combinatorial and computational properties.
- Lately, we have been thinking about *cubical* structure.

The Cubical Perspective

Several cubical structures have been proposed
as a basis for models of higher-dimensional type theory.

We survey some of their features.

Abstract Cubes

A **cube category** is a symmetric monoidal category with a distinguished object, the **abstract interval**, I.

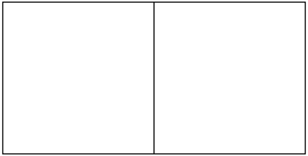
In a cube category, $\square$, for each $n \in \mathbb{N}$, we have an **abstract n-dimensional cube**, $[n] := \underbrace{\mathrm{I} \otimes \cdots \otimes \mathrm{I}}_{n}$.

0-Dimensional Cube (point)

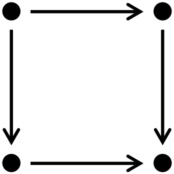
$[0]$

1-Dimensional Cube (interval)

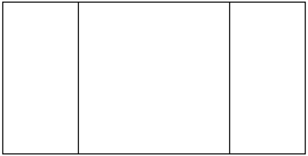
$[1]$



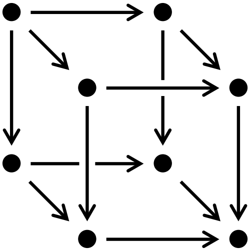
2-Dimensional Cube (square)



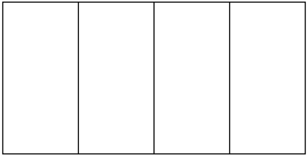
$[2]$



3-Dimensional Cube (cube)



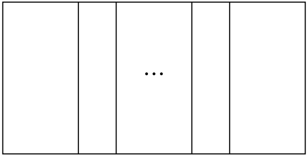
$[3]$



n-Dimensional Cube

???

$[n]$

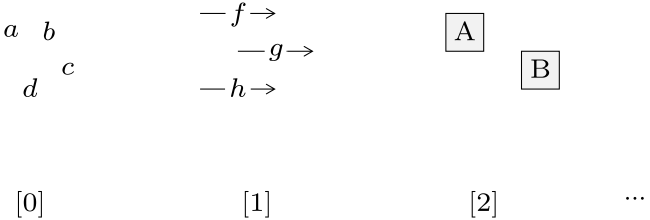


Cubiness

We seek an equational presentation of cubes
so we can describe cubes of any dimension
and the relationships between them.

Cubical Sets

A **cubical set** is a *presheaf* on a cube category:



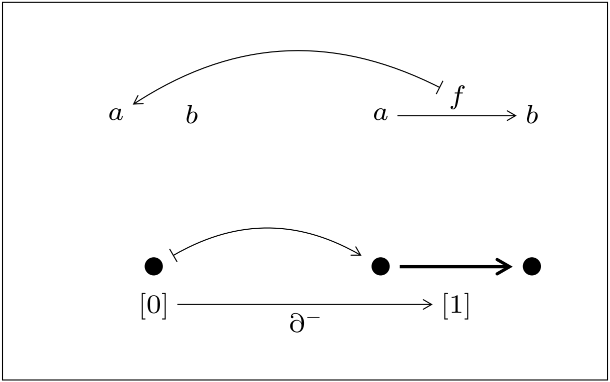
- The cubes we are interested in reside in the *fibers*, sorted by dimension.
- Maps between abstract cubes determine *contravariant* functions describing relationships between cubes.

Boundary Maps

An abstract interval has two distinguishable boundary points.

This gives us a notion of a path.

$$\partial^- , \partial^+ : \Box\,([0] \to [1])$$



Boundary Maps

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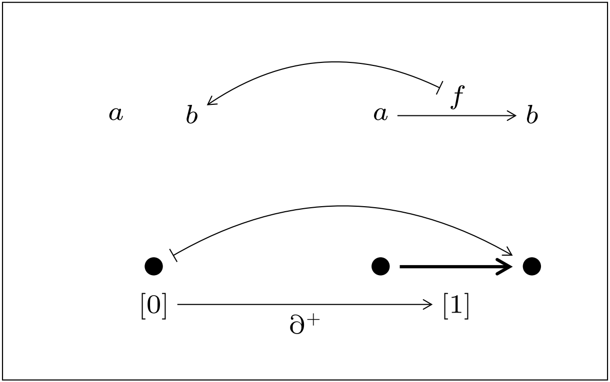
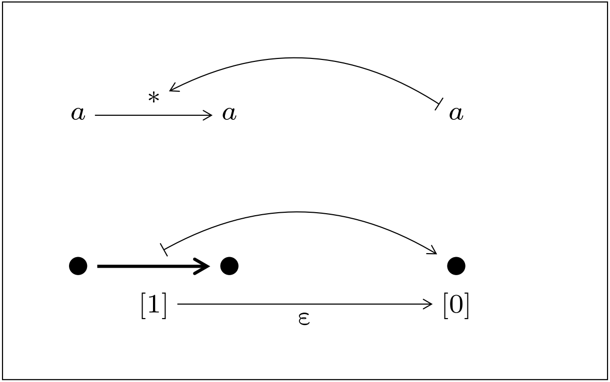


Degeneracies

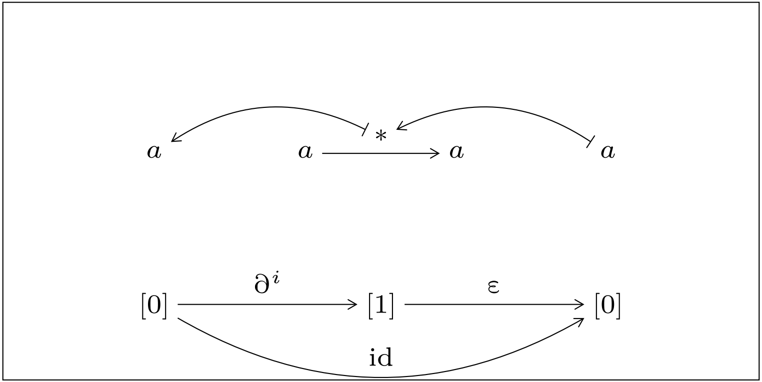
Represent the idea of a trivial path:

$$\varepsilon : \square([1] \to [0])$$



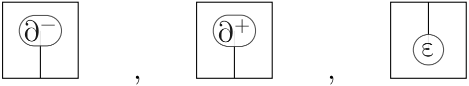
Boundary-Degeneracy Laws

$$\partial^i \cdot \varepsilon = \mathrm{id}([0])$$

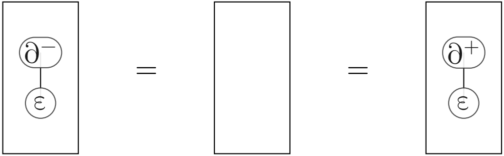


$\square(\partial, \varepsilon)$

generators



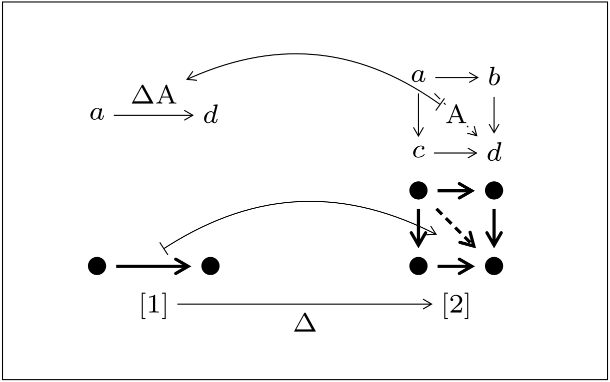
relations



Diagonal Maps

Represent the idea of a path cutting through the middle of a square:

$$\Delta : \square\,([1] \to [2])$$

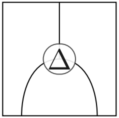


$\square(\partial\ ,\Delta)$

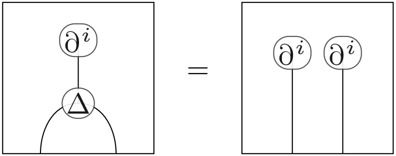
So far, the diagonal is under-specified:
we don't say *how* to cut through the middle of a square.

But there is still something that we know for certain: its boundary.

generator



relations



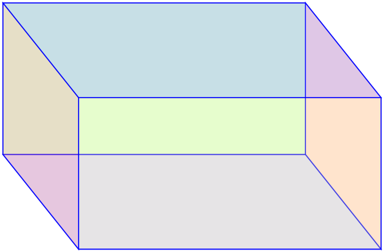
Symmetrical Diagonals

If the diagonal cuts through the square in “a straight line”
then we get more laws:

diagonal-diagonal law

$$\Delta \cdot (\Delta \otimes [1]) = \Delta \cdot ([1] \otimes \Delta)$$

represents cutting through the middle of a 3-cube.



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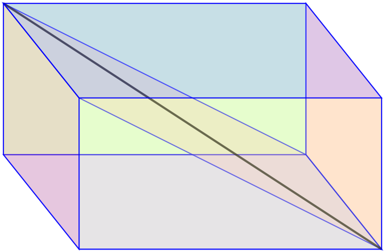

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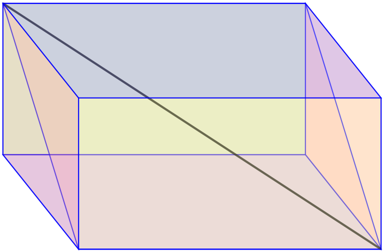


Symmetrical Diagonals

Also, putting the interval in the diagonal of the square
and then squishing the square back into the interval
along either dimension is identity:

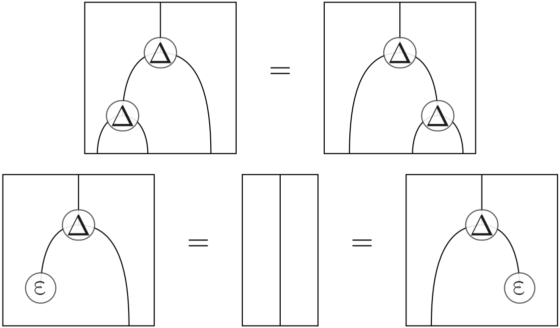
diagonal-degeneracy laws

$$\Delta \cdot (\varepsilon \otimes [1]) = \mathrm{id}([1]) = \Delta \cdot ([1] \otimes \varepsilon)$$

$\square(\varepsilon\ ,\Delta)$

You may recognize these as the *comonoid* laws:

relations



If we extend this comonoid structure *naturally* to all $[n]$,
then the monoidal structure becomes **cartesian**.

Cartesian Cubical Sets

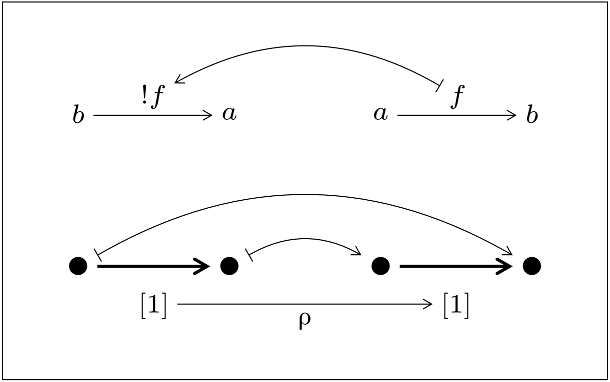
Cartesian cubical sets have several good properties, eg:

- It is a *strict test category* (has the “right homotopy theory”).
- Contexts of dimension variables behave *structurally* (admit exchange, weakening and contraction).

Reversals

Represent the idea of following a path *backwards*:

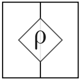
$$\rho : \Box\,([1] \to [1])$$



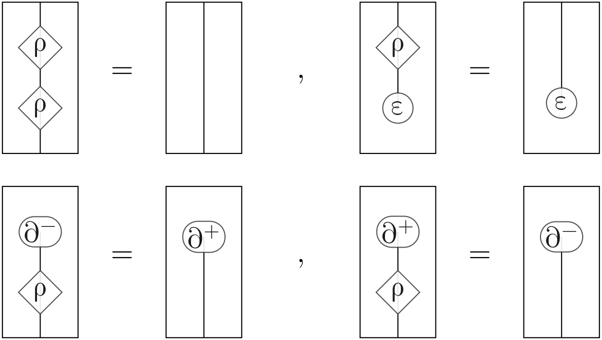
$\square(\partial\ , \varepsilon\ , \rho)$

The theory $\square(\partial\ , \varepsilon)$ plus:

generator



relations

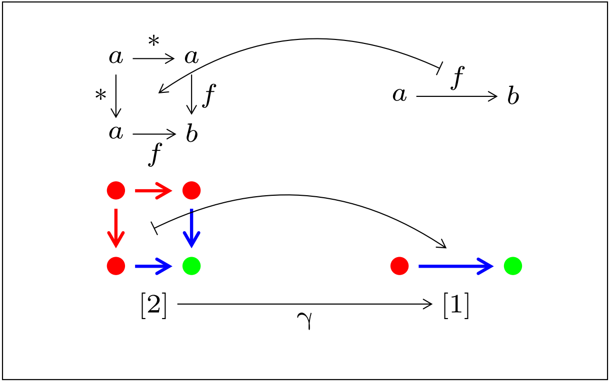


Connections

Represent another kind of degeneracy, identifying *adjacent*, rather than *opposite*, sides of an abstract cube.

They collapse a square to an interval, like a folding paper fan:

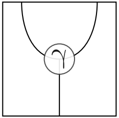
$$\gamma : \square([2] \to [1])$$



$\square(\partial , \varepsilon , \gamma)$

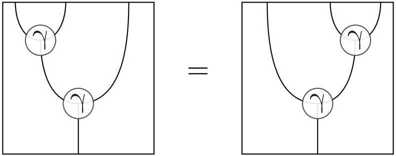
The theory $\square(\partial , \varepsilon)$ plus:

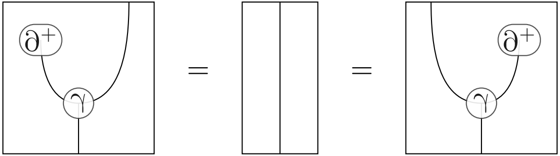
generator



relations

(γ , ∂^+) forms a *monoid*:

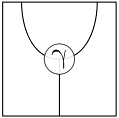




$\square(\partial\,,\varepsilon\,,\gamma)$

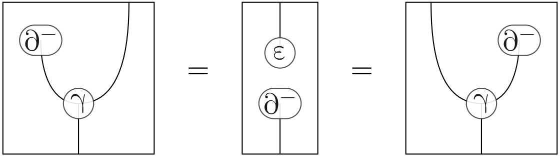
The theory $\square(\partial\,,\varepsilon)$ plus:

generator



relations

∂^- is an *absorbing element* (zero) for this monoid:



$\square(\partial\ ,\varepsilon\ ,\gamma)$

The theory $\square(\partial\ ,\varepsilon)$ plus:

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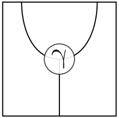
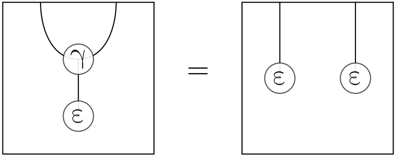


relations

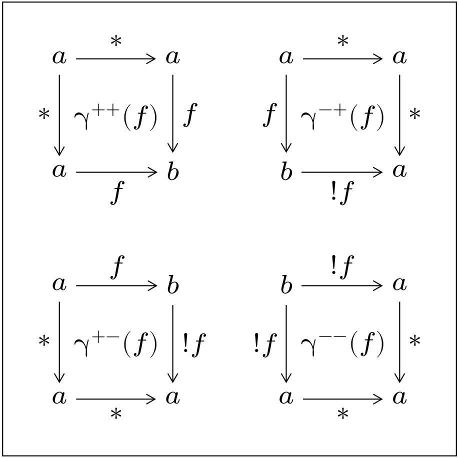
ε is a *morphism* for this monoid structure:



(plus boundary-degeneracy law from before)

Connections and Reversals

Using reversal, we get three more connections, one for “folding” the square at each of its corners:



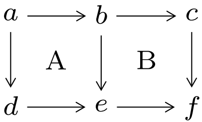
Composition

Fancier structures, such as **cubical groupoids**, extend cubical sets with a **composition structure**.

E.g. $f +_x g$:

$$a \xrightarrow{\;f\;} b \xrightarrow{\;g\;} c$$

E.g. $\mathrm{A} +_x \mathrm{B}$:



(laws available but elided)

Subdivision

In some cases, we may be able to *subdivide* cubes in a canonical way.

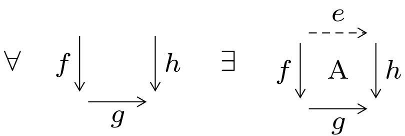
E.g. *padding*:

$$a \xrightarrow{*} a \xrightarrow{f} b \xrightarrow{*} b$$

$\square$	$\Vert$	$\square$
$=$	A	$=$
$\square$	$\Vert$	$\square$

Box Filling

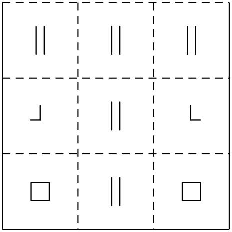
Cubical sets with the **box-filling property** are called “Kan”:



Kan cubical sets that are *uniform* with respect to degeneracies are important for interpreting higher-dimensional type theories.

Constructive Box Filling

With reversals and connections, we can constructively fill padded boxes in a cubical set.



A right adjoint to subdivision then lets us fill boxes in the **fibrant replacement** of a cubical set.

Thanks!