Categorical Semantics for Proof Search in Logic Programming

(a magic trick in three acts)

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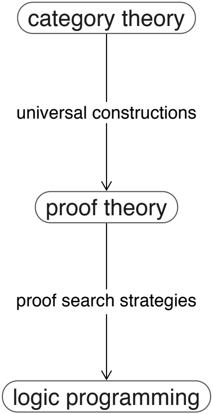
The Big Picture

What does “$\dashv$” have to do with “$\vdash$”?



Act 1: in which the magician shows you something ordinary

- Derivation Systems in Proof Theory
  - Natural Deduction
  - Sequent Calculus
- Constructions in Category Theory

Derivation Trees

In proof theory formalized inferences may be encoded as **derivation** trees, constructed inductively from primitive **inference rules**:

$$\dfrac{\overbrace{P_1 \quad \cdots \quad P_n}^{\text{collection of premises}}}{\underbrace{Q}_{\text{single conclusion}}}\ \text{[rule name]}$$

Terminology:

- An inference rule with no premises is an **axiom**.
- The conclusion at the root of a derivation is its “goal” or **end-formula**.
- The premises at the leaves are its “assumptions” or **frontier**.
- A derivation with empty frontier is a **proof**.
- A derivation with no inferences is an **identity derivation**.

Gentzen Systems

In the 1930s, Gentzen devised two types of derivation system to formalize logical proofs [Gen35]:

Natural Deduction
: A one-dimensional system:
rules represent inferences between propositions.

Sequent Calculus
: A two-dimensional system:
rules represent inferences between inferences between propositions.

Intuitionistic Natural Deduction (NJ)

Each logical connective has a set of introduction and elimination rules:

introduction rules

- have the connective principal in the *conclusion*,
- determine from what *evidence* such a proposition may be inferred:

$$\frac{\overbrace{P_1 \quad \cdots \quad P_n}^{\text{immediate evidence}}}{Q(*)} \; *+$$

elimination rules

- have the connective principal in the *major premise*,
- determine what *consequences* may be inferred from such a proposition:

$$\frac{\overbrace{P(*)}^{\text{major premise}} \qquad \overbrace{P_{m_1} \quad \cdots \quad P_{m_n}}^{\text{minor premises}}}{\underbrace{Q}_{\text{immediate consequence}}} \; *-$$

Intuitionistic Natural Deduction (NJ)

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introduction rules

- have the connective principal in the *conclusion*,
- determine from what *evidence* such a proposition may be inferred:

example: $$\frac{A \qquad B}{A \wedge B}\ \wedge{+}$$

elimination rules

- have the connective principal in the *major premise*,
- determine what *consequences* may be inferred from such a proposition:

example: $$\frac{A \wedge B}{A}\ \wedge{-}_1 \qquad \frac{A \wedge B}{B}\ \wedge{-}_2$$

Hypothetical Judgement

Part of the meta-theory of natural deduction is **hypothetical judgement**, which allows rules to have assumptions that are *local* to certain subderivations.

example:
$$\dfrac{A \vee B \qquad \dfrac{\dfrac{[A]}{\mathcal{D}_1}}{C} \qquad \dfrac{\dfrac{[B]}{\mathcal{D}_2}}{C}}{C}\ \vee-$$

These **local assumptions** don't enter the *frontier*.

The inference rules of natural deduction are given in appendix 1.

Harmony of the Connectives

The inference rules for the connectives are finely balanced, possessing a **harmony** of two parts [Dum91]:

local soundness

- elimination rules are no stronger than introduction rules
  "you can't get out more than you put in",
- witnessed by **local reductions** that remove unnecessary detours,
- determines a **computation principle** for the connective (β-reduction).

local completeness

- elimination rules are no weaker than introduction rules
  "you can get back out all that you put in",
- witnessed by **local expansions** that introduce canonical forms,
- determines a **representation principle** for the connective (η-expansion).

This harmony acts as an information-theoretic *conservation law*.

Harmony of the Connectives

The inference rules for the connectives are finely balanced, possessing a **harmony** of two parts [Dum91]:

local soundness

example:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A_1} \quad \dfrac{\mathcal{D}_2}{A_2}}{A_1 \wedge A_2}\wedge+}{A_i}\wedge-_i \quad \overset{\wedge>}{\longmapsto} \quad \dfrac{\mathcal{D}_i}{A_i}$$

local completeness

example:

$$\dfrac{\mathcal{E}}{A \wedge B} \quad \overset{\wedge<}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}}{A \wedge B}}{A}\wedge-_1 \quad \dfrac{\dfrac{\mathcal{E}}{A \wedge B}}{B}\wedge-_2}{A \wedge B}\wedge+$$

The derivation conversions witnessing harmony are given in appendix 1.

Unique Normalization of Derivations

- Because of *hypothetical judgement* we also need another kind of derivation transformation, called a **permutation conversion** for the connectives $\{\vee, \exists, \bot\}$ (see appendix 1).
- Under the relation on derivations generated by the local reductions and permutation conversions, every derivation has a *unique normal form*. [Pra65; Gir72]
- Locally expanding assumptions yields the β-normal–η-long forms. These are the canonical **normal forms** for natural deduction derivations.

Intuitionistic Sequent Calculus (LJ)

A **sequent** is an expression of the form,

$$\Gamma \implies A$$

- Γ is a collection of propositions, called the "context" or **antecedent**.
- A is a single proposition, called the "goal" or **succedent**.
- Intuitively, this sequent expresses the inference of A from Γ.

There are of two kinds of inference rules: structural and logical.

The **structural rules** tell us about the meta-theory of the logic by determining how contexts affect inference.

Sequent Calculus Logical Rules

Each connective has a set of right and left **logical rules**. These act on a proposition in the conclusion with the given connective principal:

right rules
act on the *succedent* of the conclusion.

$$\frac{\Gamma_1 \implies A_1 \qquad \cdots \qquad \Gamma_n \implies A_n}{\Gamma \implies A(*)} *R$$

left rules
act on a member of the *antecedent* of the conclusion.

$$\frac{\Gamma_1 \implies A_1 \qquad \cdots \qquad \Gamma_n \implies A_n}{\Gamma, A(*) \implies B} *L$$

The proposition $A(*)$ is the **principal formula** of the rule.
It is the only formula that is decomposed by the rule.

Sequent Calculus Logical Rules

Each connective has a set of right and left **logical rules**. These act on a proposition in the conclusion with the given connective principal:

right rules
: act on the *succedent* of the conclusion.

example: $$\frac{\Gamma, A \implies B}{\Gamma \implies A \supset B} \supset R$$

left rules
: act on a member of the *antecedent* of the conclusion.

example: $$\frac{\Gamma \implies A \qquad \Gamma, B \implies C}{\Gamma, A \supset B \implies C} \supset L$$

The rules of sequent calculus are given in appendix 2.

Sequent Calculus and Natural Deduction

These derivation systems are closely related:

Sequent calculus $\left\{ \begin{array}{c} \textit{right} \\ \textit{left} \end{array} \right\}$ rules correspond respectively to

natural deduction $\left\{ \begin{array}{c} \textit{introduction} \\ \textit{elimination} \end{array} \right\}$ rules.

This correspondence is defined by the **Prawitz translation** [Pra65].

Sequent calculus *proofs* are instructions for building a natural deduction *derivations*.

So why use sequent calculus?

It is better for *proof search*:

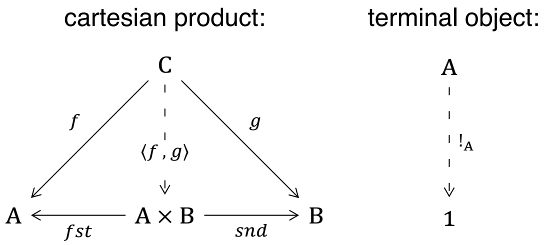
context is *local* (i.e. in the antecedents), so derivations may be constructed *unilaterally*, from root to leaves.

- Derivation Systems in Proof Theory
- Constructions in Category Theory
  - Bicartesian Closed Categories
  - Indexed Categories
  - Adjunctions

Bicartesian Closed Categories

A **bicartesian closed category** is one with:

- finite products
- finite coproducts
- exponentials



A category with just finite products is a **cartesian category**.

Bicartesian Closed Categories

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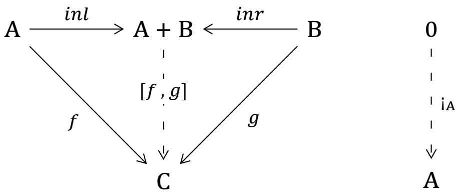
binary coproduct: initial object:



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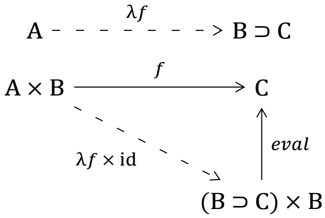
exponential (currying):



($B \supset C$ is also written "$[B\,,C]$" or "C^B")

Bicartesian Closed Categories

A **bicartesian closed category** is one with:

- finite products
- finite coproducts
- exponentials

The 2-category of bicartesian closed categories, functors and natural transformations is called “BCC”.

BCC functors preserve BCC structure.

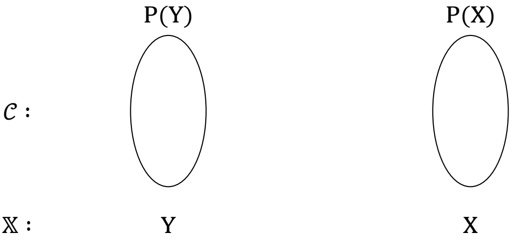
Indexed Categories

An **indexed category** is a contravariant functor from a **base category** to a 2-category of categories:

$$P : \mathbb{X}^{\circ} \longrightarrow \mathcal{C}$$

taking:

- each object of the base to its **fiber**
- each arrow of the base to a **reindexing functor**



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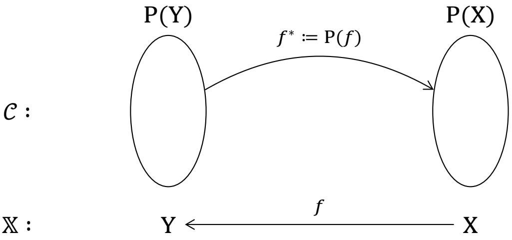

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We will be interested in indexed bicartesian closed categories with cartesian base:

$$\mathrm{P} : \mathbb{X}^{\circ} \longrightarrow \mathrm{BCC}$$

Adjunctions

An adjunction is an extremely general categorical construction with several equivalent characterizations.

Adjoints are everywhere.

- Saunders Mac Lane [Mac98]

Adjunctions by Universal Properties

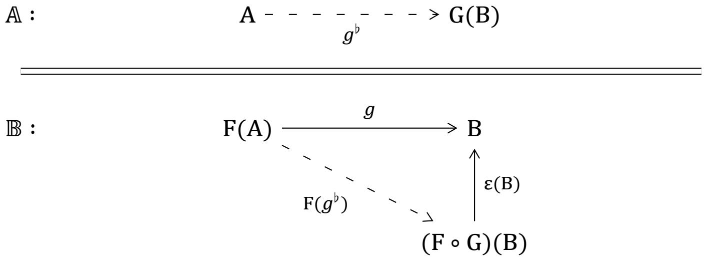
Antiparallel functors $\mathrm{F} : \mathbb{A} \longrightarrow \mathbb{B}$ and $\mathrm{G} : \mathbb{B} \longrightarrow \mathbb{A}$ form an **adjunction** "$\mathrm{F} \dashv \mathrm{G}$" with F the **left adjoint** and G the **right adjoint** if:

- **universal property of the counit**

  There is a natural transformation, called the **counit**, $\varepsilon : \mathrm{G} \cdot \mathrm{F} \longrightarrow \mathrm{id}_{\mathbb{B}}$ such that:

$$\forall\, g : \mathbb{B}\,(\mathrm{F}(\mathrm{A}) \to \mathrm{B})\,.\ \exists!\, g^{\flat} : \mathbb{A}\,(\mathrm{A} \to \mathrm{G}(\mathrm{B}))\ \ .\ \ \mathrm{F}(g^{\flat}) \cdot \varepsilon(\mathrm{B}) = g$$

i.e.



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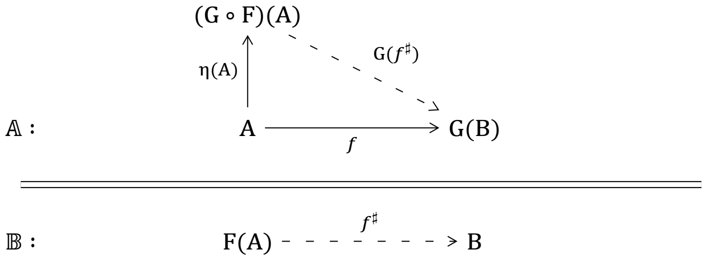
- **universal property of the unit**

  There is a natural transformation, called the **unit**, $\eta : \mathrm{id}_{\mathbb{A}} \longrightarrow \mathrm{F} \cdot \mathrm{G}$ such that:

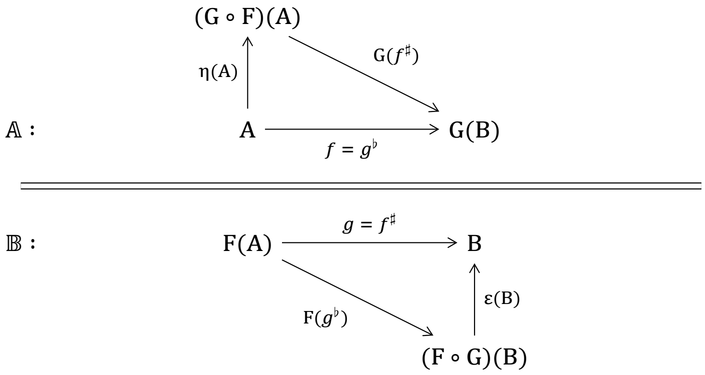
$$\forall\, f : \mathbb{A}\,(\mathrm{A} \to \mathrm{G}(\mathrm{B}))\,.\ \exists!\, f^{\sharp} : \mathbb{B}\,(\mathrm{F}(\mathrm{A}) \to \mathrm{B})\ .\ \eta(\mathrm{A}) \cdot \mathrm{G}(f^{\sharp}) = f$$

i.e.



Adjunction Summary Diagrams

Because these characterizations are equivalent, we can present an adjunction using a summary diagram:



Arrows related by the bijection are called **adjoint complements.**

Notable Adjoint Complements

We record for later use:

- the adjoint complement of a counit and unit component:

$$
\begin{array}{ll}
\mathbb{A}: & G(B) \xrightarrow{\ \mathrm{id}\ } G(B) \\
\hline\hline
\mathbb{B}: & (F \circ G)(B) \xrightarrow[\ \varepsilon\ ]{} B
\end{array}
\qquad
\begin{array}{ll}
\mathbb{A}: & A \xrightarrow{\ \eta\ } (G \circ F)(A) \\
\hline\hline
\mathbb{B}: & F(A) \xrightarrow[\ \mathrm{id}\ ]{} F(A)
\end{array}
$$

- the naturality of the adjoint complement bijection in the domain and codomain coordinate:

$$
\begin{array}{ll}
\mathbb{A}: & A' \xrightarrow{\ a\ } A \xrightarrow{\ g^{\flat}\ } G(B) \\
\hline\hline
\mathbb{B}: & F(A') \xrightarrow[\ F(a)\ ]{} F(A) \xrightarrow[\ g\ ]{} B
\end{array}
\qquad
\begin{array}{ll}
\mathbb{A}: & A \xrightarrow{\ f\ } G(B) \xrightarrow{\ G(b)\ } G(B') \\
\hline\hline
\mathbb{B}: & F(A) \xrightarrow[\ f^{\sharp}\ ]{} B \xrightarrow[\ b\ ]{} B'
\end{array}
$$

Act 2: in which the magician takes the ordinary something and makes it do something extraordinary

Categorical Logic

The basic idea:

We give an **interpretation of a logical language** $\mathcal{L}$ in a category $\mathbb{C}$:

$$[\![-]\!] : \mathcal{L} \longrightarrow \mathbb{C}$$

by sending *propositions* to *objects* and valid *inferences* to *arrows* between them:

$$\Gamma \vdash_i A \qquad \longmapsto \qquad [\![i]\!] : \mathbb{C}\left([\![\Gamma]\!] \to [\![A]\!]\right)$$

To determine what sort of category $\mathbb{C}$ should be, we examine the structure of $\mathcal{L}$ and look for universal constructions to interpret its features.

Interpreting Propositional Connectives

- **Propositional logic** is freely generated from atomic propositions by the **propositional connectives**:

$$\{\wedge, \vee, \top, \bot, \supset\}$$

- A well-known lattice-theoretic model is that of a *Heyting algebra* (i.e. bicartesian closed poset).
- Allowing parallel arrows gives an **interpretation of propositional connectives** in *bicartesian closed categories*:

connective		interpretation	
conjunction	($\wedge$)	cartesian product	($\times$)
disjunction	($\vee$)	coproduct	($+$)
truth	($\top$)	terminal object	(1)
falsehood	($\bot$)	initial object	(0)
implication	($\supset$)	exponential	($\supset$)

Interpreting Propositional Contexts

The logical assumptions of an inference are its **propositional context.**

- Inferences may have multiple (including possibly zero) assumptions.
- Assumptions may be used more than once, or not at all.

This suggests that we define the **interpretation of propositional contexts** inductively by *finite products*:

$$
\begin{aligned}
[\![\overbrace{\emptyset}^{\text{empty context}}]\!] &:= 1 \\
[\![\underbrace{\Gamma , A}_{\text{extended context}}]\!] &:= [\![\Gamma]\!] \times [\![A]\!]
\end{aligned}
$$

Interpreting Types

For first-order logic we must introduce terms and predicates as well.

We want our language of terms to be *typed*.

- We begin from an arbitrary set of **atomic types**, $\mathcal{T}$.
- For simplicity, these are the only types we consider.

An **interpretation of atomic types** in a *cartesian category* $\mathbb{C}$ is any function mapping atomic types to objects: for $\mathrm{X} \in \mathcal{T}$,

$$[\![\mathrm{X}]\!] \quad : \quad \mathbb{C}$$

Interpreting Typing Contexts

A **typing context** is a sequence of types, or equivalently, a collection of distinct typed variables[1]:

$$\Phi \quad = \quad x_1 : \mathrm{X}_1 \ , \ \cdots \ , \ x_n : \mathrm{X}_n$$

We define the **interpretation of typing contexts** inductively by *finite products*:

$$\begin{aligned} [\![\overbrace{\emptyset}^{\text{empty context}}]\!] \quad &:= \quad 1 \\ [\![\underbrace{\Phi , x : \mathrm{X}}_{\text{extended context}}]\!] \quad &:= \quad [\![\Phi]\!] \times [\![\mathrm{X}]\!]_{\mathcal{T}} \end{aligned}$$

We can forget a variable in scope using a **single omission**:

$$\hat{x} \quad : \quad \Phi , x : \mathrm{X} \longmapsto \Phi$$

It is interpreted by a complement-projection:

$$[\![\hat{x}]\!] \quad := \quad \pi : [\![\Phi , x : \mathrm{X}]\!] \longrightarrow [\![\Phi]\!]$$

[1] up to renaming

Interpreting Function Symbols

A signature for a typed term language has a collection of typed **function symbols**, $\mathcal{F}$:

$$f \quad \in \quad \mathcal{F}(\overbrace{Y_1, \cdots, Y_n}^{\text{argument context}}; \underbrace{X}_{\text{result type}})$$

Applying f to terms of types $\vec{Y}$ yields a **term** of type X.

An **interpretation of function symbols** in a *cartesian category* is any function mapping function symbols to arrows in the corresponding hom sets:

$$[\![f]\!] \quad : \quad [\![Y_1, \cdots, Y_n]\!]_{\mathcal{T}} \longrightarrow [\![X]\!]_{\mathcal{T}}$$

Terms may be **open**, i.e. contain free variables.

We express this with a **term in context**:

$$\underbrace{\Phi}_{\text{typing context}} \mid \overbrace{t}^{\text{term}} : \underbrace{X}_{\text{type}}$$

Interpreting Terms in Context

We define the **interpretation of terms** inductively by *precomposition*.

$$[\![\Phi \mid t : \mathrm{X}]\!] \quad : \quad [\![\Phi]\!]_{\mathcal{T}} \longrightarrow [\![\mathrm{X}]\!]_{\mathcal{T}}$$

lifted variable: for variable $x \notin \Phi$,

$$[\![\Phi\,, x : \mathrm{X} \mid x : \mathrm{X}]\!] \;\coloneqq\; \pi_x$$

applied function symbol: for function symbol $f \in \mathcal{F}(\mathrm{Y}_1\,, \cdots, \mathrm{Y}_n\,; \mathrm{X})$
and terms $\Phi \mid t_1 : \mathrm{Y}_1\,, \cdots, t_n : \mathrm{Y}_n$,

$$[\![\Phi \mid f(t_1\,, \cdots, t_n) : \mathrm{X}]\!] \;\coloneqq\; \langle [\![t_1]\!]\,, \cdots, [\![t_n]\!] \rangle \cdot [\![f]\!]_{\mathcal{F}}$$

context extension: for term $\Phi \mid t : \mathrm{X}$ and “dummy” variable $x \notin \Phi$,

$$[\![\Phi\,, x : \mathrm{X} \mid t : \mathrm{X}]\!] \;\coloneqq\; [\![\hat{x}]\!]_{\mathcal{T}} \cdot [\![t]\!]$$

substitution: for terms $\Phi\,, y : \mathrm{Y} \mid t : \mathrm{X}$ and $\Phi \mid s : \mathrm{Y}$,

$$[\![\Phi \mid t[y \mapsto s] : \mathrm{X}]\!] \;\coloneqq\; [\![\,[y \mapsto s]\,]\!] \cdot [\![t]\!]$$

where we define the interpretation of a **single substitution** as:

$$[\![\,[y \mapsto s]\,]\!] \quad \coloneqq \quad \langle \mathrm{id}_{[\![\Phi]\!]_{\mathcal{T}}}\,, [\![s]\!] \rangle$$

Interpreting Relation Symbols

A signature for a typed predicate language has a collection of typed **relation symbols**, $\mathcal{R}$:

$$\mathrm{R} \quad \in \quad \mathcal{R}(\overbrace{\mathrm{X}_i\,, \cdots, \mathrm{X}_n}^{\text{argument context}})$$

Applying R to terms of types $\vec{\mathrm{X}}$ yields an atomic proposition or **predicate.**

An **interpretation of relation symbols** in an *indexed category* P is any function mapping relation symbols to objects in the corresponding fibers:

$$[\![\mathrm{R}]\!] \quad : \quad \mathrm{P}([\![\mathrm{X}_1\,, \cdots, \mathrm{X}_n]\!]_{\mathcal{T}})$$

Since terms may be *open*, propositions may be too.

We express this with a **proposition in context**:

$$\underbrace{\Phi}_{\text{typing context}} \quad | \quad \overbrace{\mathrm{A}}^{\text{proposition}} \quad \mathrm{PROP}$$

Interpreting Predicates in Context

We define the **interpretation of predicates** inductively by *reindexing*.

$$[\![\Phi \mid \mathrm{A}\ \textsc{prop}]\!] \quad : \quad \mathrm{P}([\![\Phi]\!]_{\mathcal{T}})$$

applied relation symbol: for relation symbol $\mathrm{R} \in \mathcal{R}(\mathrm{Y}_1, \cdots, \mathrm{Y}_n)$
and terms $\Phi \mid t_1 : \mathrm{Y}_1, \cdots, t_n : \mathrm{Y}_n$,

$$[\![\Phi \mid \mathrm{R}(t_1, \cdots, t_n)\ \textsc{prop}]\!] \coloneqq \langle [\![t_1]\!]_{\mathcal{F}}, \cdots, [\![t_n]\!]_{\mathcal{F}} \rangle^* ([\![\mathrm{R}]\!]_{\mathcal{R}})$$

context extension: for predicate $\Phi \mid \mathrm{A}\ \textsc{prop}$ and “dummy” variable $x \notin \Phi$,

$$[\![\Phi, x : \mathrm{X} \mid \mathrm{A}\ \textsc{prop}]\!] \coloneqq [\![\hat{x}]\!]^*_{\mathcal{T}} ([\![\mathrm{A}]\!])$$

substitution: for predicate $\Phi, y : \mathrm{Y} \mid \mathrm{A}\ \textsc{prop}$ and term $\Phi \mid s : \mathrm{Y}$,

$$[\![\Phi \mid \mathrm{A}[y \mapsto s]\ \textsc{prop}]\!] \coloneqq [\![\,[y \mapsto s]\,]\!]^*_{\mathcal{F}} ([\![\mathrm{A}]\!])$$

The reindexing arrow is the same one precomposed to interpret terms.

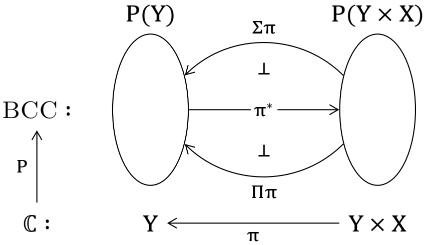
Interpreting Quantifiers

- If an *indexed category* has *adjoints* for *reindexing* by projections:

$$\Sigma\pi \dashv \pi^* \dashv \Pi\pi$$

- then we may use them as the **interpretation of quantifiers**:

$$[\![\exists x]\!] := \Sigma\pi \qquad [\![\forall x]\!] := \Pi\pi$$



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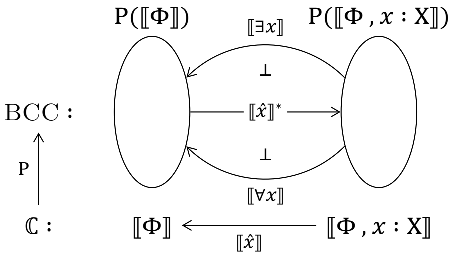

Quantifiers and Substitution

In logic, the quantifiers–like the propositional connectives–are compatible with (capture-avoiding) substitution:

$$
\begin{array}{rcl}
(\forall x : X \,.\, A)[y \mapsto t] & = & \forall x : X \,.\, (A[y \mapsto t]) \\
(\exists x : X \,.\, A)[y \mapsto t] & = & \exists x : X \,.\, (A[y \mapsto t])
\end{array}
$$

In order for their interpretations to have this property, we must impose on the **Beck-Chevalley condition**:

$$
\begin{array}{ccc}
Z \times X & \xrightarrow{f \times \mathrm{id}} & Y \times X \\
\downarrow \pi & & \downarrow \pi \\
Z & \xrightarrow[f]{} & Y
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccc}
P(Z \times X) & \xleftarrow{(f \times \mathrm{id})^*} & P(Y \times X) \\
\downarrow \text{Ծ}\pi & & \downarrow \text{Ծ}\pi \\
P(Z) & \xleftarrow[f^*]{} & P(Y)
\end{array}
$$

for $\text{Ծ} \in \{\Pi \,, \Sigma\}$.

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$$\begin{array}{rcl}(\forall x : \mathrm{X} \,.\, \mathrm{A})[y{\mapsto}t] & = & \forall x : \mathrm{X} \,.\, (\mathrm{A}[y{\mapsto}t]) \\ (\exists x : \mathrm{X} \,.\, \mathrm{A})[y{\mapsto}t] & = & \exists x : \mathrm{X} \,.\, (\mathrm{A}[y{\mapsto}t])\end{array}$$

In order for their interpretations to have this property, we must impose on the **Beck-Chevalley condition**:

$$\begin{array}{ccc} [\![\Phi , x : \mathrm{X}]\!] & \xrightarrow{[\![\,[y\mapsto t]\,]\!]} & [\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!] \\ \downarrow{\scriptstyle [\![\hat{x}]\!]} & \lrcorner & \downarrow{\scriptstyle [\![\hat{x}]\!]} \\ [\![\Phi]\!] & \xrightarrow[{[\![\,[y\mapsto t]\,]\!]}]{} & [\![\Phi , y : \mathrm{Y}]\!] \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathrm{P}([\![\Phi , x : \mathrm{X}]\!]) & \xleftarrow{[\![\,[y\mapsto t]\,]\!]^*} & \mathrm{P}([\![\Phi , x : \mathrm{X} , y : \mathrm{Y}]\!]) \\ \downarrow{\scriptstyle [\![\mho x]\!]} & & \downarrow{\scriptstyle [\![\mho x]\!]} \\ \mathrm{P}([\![\Phi]\!]) & \xleftarrow[{[\![\,[y\mapsto t]\,]\!]^*}]{} & \mathrm{P}([\![\Phi , y : \mathrm{Y}]\!]) \end{array}$$

for $\mho \in \{\forall , \exists\}$.

Interpreting First-Order Logic

We now have all the pieces we need to interpret first-order logic in categories.

- A **hyperdoctrine** is an indexed bicartesian closed category over a cartesian base with adjoints for reindexing by projections that satisfy the Beck-Chevalley condition [Law69].
- An **interpretation of a typed first-order logical language** $\mathcal{L}$ with signature $(\mathcal{T}, \mathcal{F}, \mathcal{R})$ in a hyperdoctrine $\mathrm{P} : \mathbb{C}^\circ \longrightarrow \mathrm{BCC}$ is determined by $[\![-]\!]_{\mathcal{T}}$, $[\![-]\!]_{\mathcal{F}}$ and $[\![-]\!]_{\mathcal{R}}$ together with the given interpretations of the *propositional connectives* and *quantifiers*.
- We are especially interested in **freely-generated interpretations**. These have only those objects, arrows and equations required by the defining categorical constructions. In this case, we write:

$$\mathrm{P{\scriptstyle ROP}}_{\mathcal{L}} : \mathrm{T{\scriptstyle YPE}}_{\mathcal{L}}{}^\circ \longrightarrow \mathrm{BCC}$$

and suppress the interpretation brackets, "$[\![-]\!]$".

- Categorical Logic
- Natural Deduction by Adjunction
  - Inference Rules
  - Derivation Conversions
  - Genericity of Free Hyperdoctrine Semantics

The Connectives by Adjunction

All of the universal constructions interpreting the connectives of intuitionistic first-order logic are definable by adjunctions:

- for the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$,

$$\underline{1} + \underline{2} \quad \dashv \quad \Delta \quad \dashv \quad \underline{1} \times \underline{2}$$

- for the unique functor $! : \mathbb{C} \longrightarrow \mathbb{1}$,

$$0 \quad \dashv \quad ! \quad \dashv \quad 1$$

- for any $B : \mathbb{C}$,

$$- \times B \quad \dashv \quad B \supset -$$

- for a projection π,

$$\Sigma\pi \quad \dashv \quad \pi^* \quad \dashv \quad \Pi\pi$$

The Connectives by Adjunction

All of the universal constructions interpreting the connectives of intuitionistic first-order logic are definable by adjunctions:

- for the diagonal functor $\Delta : \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$,

$$[\![\frac{1}{} \vee \frac{2}{}]\!] \quad \dashv \quad \Delta \quad \dashv \quad [\![\frac{1}{} \wedge \frac{2}{}]\!]$$

- for the unique functor $! : \mathbb{C} \longrightarrow \mathbb{1}$,

$$[\![\bot]\!] \quad \dashv \quad ! \quad \dashv \quad [\![\top]\!]$$

- for any $\mathrm{B} : \mathbb{C}$,

$$[\![- \wedge \mathrm{B}]\!] \quad \dashv \quad [\![\mathrm{B} \supset -]\!]$$

- for a projection π,

$$[\![\exists x]\!] \quad \dashv \quad [\![\hat{x}]\!]^{*} \quad \dashv \quad [\![\forall x]\!]$$

Natural Deduction by Adjunction

Remarkably, we can reconstruct the derivation system of natural deduction *uniformly* from these adjunctions.

A single categorical construction generates the entire proof theory!

The adjunction-based perspective provides insight as well as concision:

- The connectives are partitioned into two sets by their **chirality**: whether they are characterized by a right or left adjoint functor. We call them **right connectives** $\{\wedge, \top, \supset, \forall\}$ and **left connectives** $\{\vee, \bot, \exists\}$.
- Connectives can be defined on derivations as well as on propositions (by functoriality).
- *Permutation conversions* can be defined for right as well as left connectives (by naturality).
- The non-invertible quantifier rules can be decomposed into a strictly logical part and a substitution–very useful for *proof search*.

Natural Deduction by Adjunction

Theorem

The adjunction-based interpretation of the connectives extends to an interpretation of the inference rules and derivation conversions of natural deduction in a uniform way:
For right connectives,
introduction rules:
adjoint complement operation $(-^\flat)$,
elimination rules:
adjunction counit (ε),
local reductions:
factorization in the universal property of the counit $(\mathrm{F}(\mathcal{D}^\flat)\cdot\varepsilon = \mathcal{D})$,
permutation conversions: *(implicit in Gentzen's syntax)*
naturality of the adjoint complement bijection in the domain coordinate $(\mathcal{E}\cdot\mathcal{D}^\flat = (\mathrm{F}(\mathcal{E})\cdot\mathcal{D})^\flat)$,
local expansions:
identity maps on right adjoint images are adjoint complements of counit components $(\mathrm{id}_\mathrm{G} = \varepsilon^\flat)$.

Natural Deduction by Adjunction

Theorem

The adjunction-based interpretation of the connectives extends to an interpretation of the inference rules and derivation conversions of natural deduction in a uniform way:

For left connectives,

introduction rules:

adjunction unit (η),

elimination rules:

adjoint complement operation ($-^{\sharp}$),

local reductions:

factorization in the universal property of the unit ($\eta \cdot \mathrm{G}(\mathcal{D}^{\sharp}) = \mathcal{D}$),

permutation conversions:

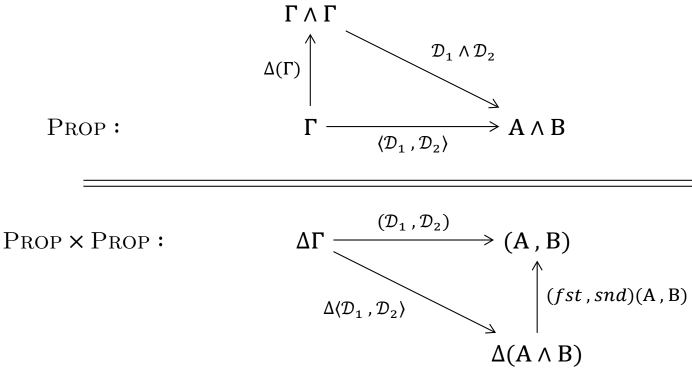
naturality of the adjoint complement bijection in the codomain coordinate ($\mathcal{D}^{\sharp} \cdot \mathcal{E} = (\mathcal{D} \cdot \mathrm{G}(\mathcal{E}))^{\sharp}$),

local expansions:

identity maps on left adjoint images are adjoint complements of unit components ($\mathrm{id}_{\mathrm{F}} = \eta^{\sharp}$).

Case: Conjunction

We can summarize the adjunction for *conjunction* ($\Delta \dashv [\![\wedge]\!]$) with the diagram:



"PROP" is shorthand for PROP(Φ) for arbitrary typing context Φ.

Introduction Rule

The bijection of this adjunction lets us swap a derivation in $\textsc{Prop} \times \textsc{Prop}$ from common assumptions for a one in $\textsc{Prop}$ to a conjunction:

$$(\mathcal{D}_1, \mathcal{D}_2) : \Delta\Gamma \longrightarrow (A, B) \quad \overset{\_^\flat}{\longmapsto} \quad \langle \mathcal{D}_1, \mathcal{D}_2 \rangle : \Gamma \longrightarrow A \wedge B$$

In derivation notation:

$$\begin{array}{c}\Gamma\\\hline \mathcal{D}_1\\\hline A\end{array} \qquad \begin{array}{c}\Gamma\\\hline \mathcal{D}_2\\\hline B\end{array} \quad \overset{\_^\flat}{\longmapsto} \quad \dfrac{\Gamma \qquad \begin{array}{c}\overline{\Gamma}\\\hline \mathcal{D}_1\\\hline A\end{array} \qquad \begin{array}{c}\overline{\Gamma}\\\hline \mathcal{D}_2\\\hline B\end{array}}{A \wedge B} \wedge +^\star$$

This rule is *interchangeable* with Gentzen's rule when its assumptions are made explicit, except that we *account* for their duplication:

$$\dfrac{\begin{array}{c}\Gamma\\\hline \mathcal{D}_1\\\hline A\end{array} \qquad \begin{array}{c}\Gamma\\\hline \mathcal{D}_2\\\hline B\end{array}}{A \wedge B} \wedge +$$

Elimination Rule

The counit of this adjunction is the ordered pair of projections:

$$\varepsilon \quad = \quad (fst\,, snd)$$

As a pair of inference rules, these are exactly the elimination rules for ∧:

$$A \qquad B \qquad \stackrel{\varepsilon}{\longmapsto} \qquad \frac{A \wedge B}{A}\ \wedge\!-_1 \quad \frac{A \wedge B}{B}\ \wedge\!-_2$$

So the counit is an inference rule in the product category, $\textsc{Prop} \times \textsc{Prop}$.

Local Reduction

The factorization in the *universal property of the counit*:

$$\Delta\langle \mathcal{D}_1 , \mathcal{D}_2 \rangle \cdot (fst , snd) \quad = \quad (\mathcal{D}_1 , \mathcal{D}_2)$$

translated into derivation notation:

$$\dfrac{\dfrac{\Gamma \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_1}}{A} \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_2}}{B}}{A \wedge B}\ \wedge+^{\star}}{A}\ \wedge-_1 \qquad \dfrac{\dfrac{\Gamma \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_1}}{A} \qquad \dfrac{\dfrac{\overline{\Gamma}}{\mathcal{D}_2}}{B}}{A \wedge B}\ \wedge+^{\star}}{B}\ \wedge-_2 \qquad \overset{\wedge>^{\star}}{\longmapsto} \qquad \dfrac{\dfrac{\Gamma}{\mathcal{D}_1}}{A} \quad \dfrac{\dfrac{\Gamma}{\mathcal{D}_2}}{B}$$

gives us the ordered pair of *local reductions for conjunction*.

Permutation Conversion

The naturality of the bijection in the domain coordinate:

$$\mathcal{E} \cdot \langle \mathcal{D}_1 , \mathcal{D}_2 \rangle \quad = \quad \langle \mathcal{E} \cdot \mathcal{D}_1 , \mathcal{E} \cdot \mathcal{D}_2 \rangle$$

translated into derivation notation:

$$\dfrac{\dfrac{\dfrac{\Gamma}{\mathcal{E}}}{C} \qquad \dfrac{\dfrac{\overline{C}}{\mathcal{D}_1}}{A} \qquad \dfrac{\dfrac{\overline{C}}{\mathcal{D}_2}}{B}}{A \wedge B}\ \wedge{+}^{\star} \quad \overset{\wedge\rightleftarrows^{\star}}{\longmapsto} \quad \dfrac{\Gamma \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}}{\mathcal{E}}}{C}}{\mathcal{D}_1}}{A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}}{\mathcal{E}}}{C}}{\mathcal{D}_2}}{B}}{A \wedge B}\ \wedge{+}^{\star}$$

This says that any derivation precomposed to a $\wedge{+}^{\star}$ rule may be moved into the minor branch by duplication.

Making this duplication operation explicit sheds light on the properties of the meta-logic.

Local Expansion

The equation for adjoint complements to counit components:

$$\mathrm{id}_{-\wedge-} \quad = \quad \langle fst\,,snd \rangle$$

translated into derivation notation:

$$A \wedge B \quad \overset{\wedge<^{\star}}{\longmapsto} \quad \dfrac{A \wedge B \qquad \dfrac{\dfrac{}{A \wedge B}}{A}\,\wedge-_1 \qquad \dfrac{\dfrac{}{A \wedge B}}{B}\,\wedge-_2}{A \wedge B}\,\wedge+^{\star}$$

gives us a *local expansion for conjunction*.

We recover Gentzen's version by precomposing an arbitrary derivation and applying the permutation conversion.

□

Genericity of Free Hyperdoctrine Semantics

The adjoint-theoretic semantics generates the natural deduction proof theory. But in the case of free interpretations, the converse holds as well:

Corollary

Freely-generated hyperdoctrine interpretations *are **generic** for natural deduction: arrows in the fibers correspond precisely to equivalence classes of derivations under the conversion relations.*

So free hyperdoctrine categorical semantics essentially *is* natural deduction proof theory.

We have just made a proof-theoretic *rabbit* disappear into a categorical *hat* by waving the *magic wand* of adjunctions.

Genericity of Free Hyperdoctrine Semantics

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So free hyperdoctrine categorical semantics essentially *is* natural deduction proof theory.

We have just made a proof-theoretic *rabbit* disappear into a categorical *hat* by waving the *magic wand* of adjunctions.

Act 3: because making something disappear isn't enough – you have to bring it back

- Indexed Sequent Calculus
  - Indexed Quantifier Eigenrules
  - Indexed Quantifier Anderrules
  - Equivalence to Gentzen's System
- Proof Search Strategies for Logic Programming

Sequent Calculus in Hyperdoctrines

The **interpretation of an indexed sequent** in a hyperdoctrine is a hom set in the fibers:

$$[\![\Phi \mid \Gamma \implies A]\!] \quad := \quad P([\![\Phi]\!])\,([\![\Gamma]\!] \to [\![A]\!])$$

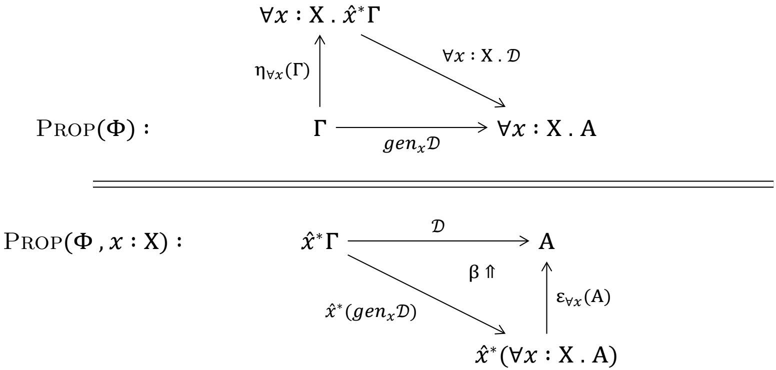
We can interpret *sequent calculus* inference rules using adjunctions as well (cf. the *Prawitz translation*).

- The bijection of adjoint complements provides interpretations for: $\left\{\begin{array}{c}\textit{right rules}\\ \textit{left rules}\end{array}\right\}$ of $\left\{\begin{array}{c}\textit{right connectives}\\ \textit{left connectives}\end{array}\right\}$. We call these **eigenrules**. They are always invertible.
- Composition with a counit or unit component provides interpretations for: $\left\{\begin{array}{c}\textit{left rules}\\ \textit{right rules}\end{array}\right\}$ of $\left\{\begin{array}{c}\textit{right connectives}\\ \textit{left connectives}\end{array}\right\}$. We call these **anderrules.**

The rules derived this way for the quantifiers lead to a formulation of sequent calculus that is *indexed by typing contexts*.

Indexed Quantification

We can summarize the adjunction for *universal quantification* ($[\![\hat{x}]\!]^* \dashv [\![\forall x]\!]$) with the diagram:



Indexed Quantifier Eigenrules

The bijection of the adjunction:

$$\frac{\textsc{Prop}(\Phi)\ (\Gamma \to \forall x : X \,.\, A)}{\textsc{Prop}(\Phi\,,\, x : X)\ (\hat{x}^{*}\Gamma \to A)}$$

yields (upside-down) an indexed sequent calculus *eigenrule*:

$$\frac{\Phi\,,\, x : X \mid \Gamma \implies A}{\Phi \mid \Gamma \implies \forall x : X \,.\, A}\ \forall R^{\star}$$

This is equivalent to the standard *right rule*:

$$\frac{\Gamma \implies A[x \mapsto e]}{\Gamma \implies \forall x : X \,.\, A}\ \forall R^{\dagger}$$

† e may not occur in the conclusion

(just let $e \coloneqq x$).

Indexed Quantifier Anderrules

$$
\begin{array}{ccccc}
\Gamma\,,\forall x : X\,.\,A & & \hat{x}^{*}\Gamma\,,\hat{x}^{*}(\forall x : X\,.\,A) & & \Gamma\,,\forall x : X\,.\,A \\
 & & \big\downarrow \mathrm{id}\,,\varepsilon_{\forall x}(A) & & \big\downarrow \mathrm{id}\,,(\varepsilon_{\forall x}(A))[x{\mapsto}t] \\
 & & \hat{x}^{*}\Gamma\,,A & & \Gamma\,,A[x{\mapsto}t] \\
 & & & & \big\downarrow \mathcal{D} \\
B & & \hat{x}^{*}B & & B \\
\Phi & \xleftarrow{\;\hat{x}\;} & \Phi\,,x : X & \xleftarrow{\;[x\mapsto t]\;} & \Phi \\
\end{array}
$$

$$\Phi \xleftarrow{\;\mathrm{id}\;} \Phi$$

The standard *left rule* for universal quantification:

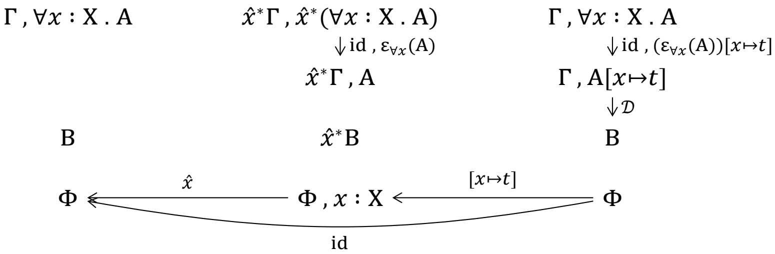
$$\dfrac{t : X \qquad \Gamma\,,A[x{\mapsto}t] \implies B}{\Gamma\,,\forall x : X\,.\,A \implies B}\ \forall L$$

can be written in the indexed sequent calculus as:

$$\dfrac{\Phi \implies t : X \qquad \Phi \mid \Gamma\,,\ A[x{\mapsto}t] \implies B}{\Phi \mid \Gamma\,,\ \forall\, x : X\,.\,A \implies B}\ \forall L$$

It corresponds to first reindexing by the term, then composing with the counit.
But what if we don't yet know which term to use?

Indexed Quantifier Anderrules



Reading composition with the counit as an inference rule gives us:

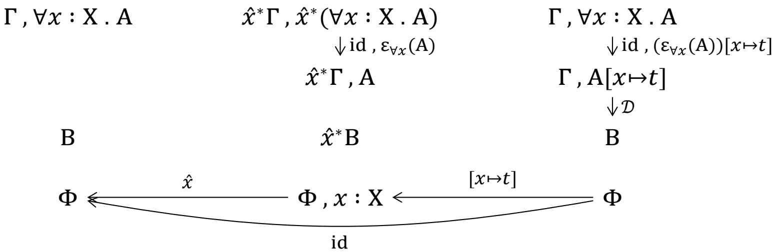
$$\frac{\Phi\,,\,x : \mathrm{X} \;\mid\; \hat{x}^*\Gamma\;,\;\mathrm{A} \;\Longrightarrow\; \hat{x}^*\mathrm{B}}{\Phi\,,\,x : \mathrm{X} \;\mid\; \hat{x}^*\Gamma\;,\;\hat{x}^*(\forall x : \mathrm{X}\,.\,\mathrm{A}) \;\Longrightarrow\; \hat{x}^*\mathrm{B}}$$

We want a substitution instance of this rule, we just don't know which one yet. But no matter the term, the conclusion will be:

$$\Phi \;\mid\; \Gamma\;,\;\forall\, x : \mathrm{X}\,.\,\mathrm{A} \;\Longrightarrow\; \mathrm{B}$$

We could use this conclusion in the rule if we knew that *some* term exists.

Indexed Quantifier Anderrules

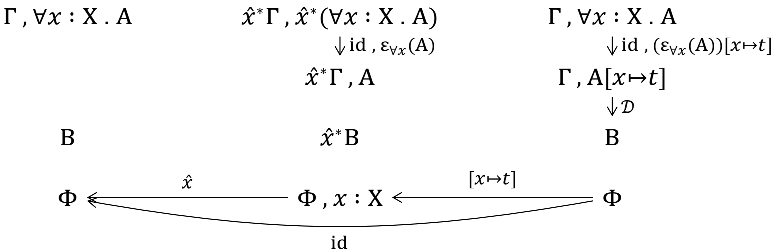


So we can add a premise to ensure that this is the case:

$$\frac{\Phi \implies \underline{x} : \mathrm{X} \qquad \Phi , \underline{x} : \mathrm{X} \mid \Gamma \, , \, \mathrm{A}[x \mapsto \underline{x}] \implies \mathrm{B}}{\Phi \mid \Gamma \, , \, \forall\, x : \mathrm{X} \, . \, \mathrm{A} \implies \mathrm{B}} \ \forall\mathrm{L}^{\star}$$

The underlining annotation reminds us that we owe a substitution for this variable.
We call this an **obligation variable**, but in the semantics it is just a context variable.

Indexed Quantifier Anderrules



We recover the standard rule by immediately choosing a term by which to reindex:

$$\dfrac{\dfrac{\Phi \;\Longrightarrow\; t : X}{\Phi \;\Longrightarrow\; \underline{x} : X}\,[\underline{x}\mapsto t] \qquad \dfrac{\Phi \mid \Gamma\ ,\ A[x\mapsto t] \;\Longrightarrow\; B}{\Phi , \underline{x} : X \mid \Gamma\ ,\ A[x\mapsto \underline{x}] \;\Longrightarrow\; B}\,[\underline{x}\mapsto t]}{\Phi \mid \Gamma\ ,\ \forall\, x : X . A \;\Longrightarrow\; B}\ \forall L^{\star}$$

But there is no reason that we need to choose the term right away.

Indexed Sequent Calculus

This motivates the **indexed sequent calculus**, with rules for

propositional connectives : as in Gentzen

quantifiers :

$$\frac{\Phi , x : X \mid \Gamma \Longrightarrow A}{\Phi \mid \Gamma \Longrightarrow \forall x : X . A} \ \forall R^\star \qquad \frac{\Phi \Longrightarrow \underline{x} : X \qquad \Phi , \underline{x} : X \mid \Gamma , A[x \mapsto \underline{x}] \Longrightarrow B}{\Phi \mid \Gamma , \forall x : X . A \Longrightarrow B} \ \forall L^\star$$

$$\frac{\Phi , x : X \mid \Gamma , A \Longrightarrow B}{\Phi \mid \Gamma , \exists x : X . A \Longrightarrow B} \ \exists L^\star \qquad \frac{\Phi \Longrightarrow \underline{x} : X \qquad \Phi , \underline{x} : X \mid \Gamma \Longrightarrow A[x \mapsto \underline{x}]}{\Phi \mid \Gamma \Longrightarrow \exists x : X . A} \ \exists R^\star$$

substitution :

$$\frac{\Phi \Longrightarrow s[\underline{x} \mapsto t] : Y}{\Phi \Longrightarrow s : Y} \ sub\ [\underline{x} \mapsto t] \qquad \frac{\Phi \mid \Gamma[\underline{x} \mapsto t] \Longrightarrow A[\underline{x} \mapsto t]}{\Phi , \underline{x} : X \mid \Gamma \Longrightarrow A} \ sub\ [\underline{x} \mapsto t]$$

Restriction:

- Substitutions may be made only for obligation variables.
- Substitution must be applied to the whole frontier to reindex a derivation.

Why Indexed Quantifier Anderrules?

Decomposing the non-invertible quantifier rules this way lets us postpone term selection until we have more information.

Example

If there's something to which everything is related, then everything is related to something:

$$\begin{array}{c}\vdots\\ \emptyset \mid \exists y : \mathrm{Y} \,.\, \forall u : \mathrm{X} \,.\, \mathrm{R}(u\,,y) \implies \forall x : \mathrm{X} \,.\, \exists v : \mathrm{Y} \,.\, \mathrm{R}(x\,,v)\end{array}$$

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$$\dfrac{\dfrac{\vdots \atop x : X, y : Y \mid \forall u : X \,.\, R(u, y) \implies \exists v : Y \,.\, R(x, v)}{x : X \mid \exists y : Y \,.\, \forall u : X \,.\, R(u, y) \implies \exists v : Y \,.\, R(x, v)}\ \exists L}{\emptyset \mid \exists y : Y \,.\, \forall u : X \,.\, R(u, y) \implies \forall x : X \,.\, \exists v : Y \,.\, R(x, v)}\ \forall R$$

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$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ x : X, y : Y \Rightarrow \underline{v} : Y\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ x : X, y : Y, \underline{v} : Y \Rightarrow \underline{u} : X\end{array} \qquad \begin{array}{c}\vdots\\ x : X, y : Y, \underline{v} : Y, \underline{u} : X \mid R(\underline{u}, y) \Rightarrow R(x, \underline{v})\end{array}}{x : X, y : Y, \underline{v} : Y \mid \forall u : X . R(u, y) \Rightarrow R(x, \underline{v})}\ \forall L}{x : X, y : Y \mid \forall u : X . R(u, y) \Rightarrow \exists v : Y . R(x, v)}\ \exists R}{x : X \mid \exists y : Y . \forall u : X . R(u, y) \Rightarrow \exists v : Y . R(x, v)}\ \exists L}{\emptyset \mid \exists y : Y . \forall u : X . R(u, y) \Rightarrow \forall x : X . \exists v : Y . R(x, v)}\ \forall R$$

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\dfrac{
  \dfrac{\dfrac{\vdots}{x:\mathrm{X}, y:\mathrm{Y} \Rightarrow y:\mathrm{Y}}}{x:\mathrm{X}, y:\mathrm{Y} \Rightarrow \underline{v}:\mathrm{Y}}\;\sigma
  \qquad
  \dfrac{
    \dfrac{\dfrac{\vdots}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \Rightarrow x:\mathrm{X}}}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \Rightarrow \underline{u}:\mathrm{X}}\;\sigma
    \qquad
    \dfrac{\dfrac{\dfrac{\vdots}{x:\mathrm{X}, y:\mathrm{Y} \mid \mathrm{R}(x,y) \Rightarrow \mathrm{R}(x,y)}}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \mid \mathrm{R}(x,y) \Rightarrow \mathrm{R}(x,\underline{v})}\;[\underline{v}\mapsto y]}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y}, \underline{u}:\mathrm{X} \mid \mathrm{R}(\underline{u},y) \Rightarrow \mathrm{R}(x,\underline{v})}\;[\underline{u}\mapsto x]
  }{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \mid \forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \mathrm{R}(x,\underline{v})}\;\forall\mathrm{L}
}{
  \dfrac{
    \dfrac{x:\mathrm{X}, y:\mathrm{Y} \mid \forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}{x:\mathrm{X} \mid \exists y:\mathrm{Y}.\,\forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}\;\exists\mathrm{L}
  }{\emptyset \mid \exists y:\mathrm{Y}.\,\forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \forall x:\mathrm{X}.\,\exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}\;\forall\mathrm{R}
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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x:\mathrm{X}, y:\mathrm{Y} \Rightarrow y:\mathrm{Y}}\;\mathit{init}}{x:\mathrm{X}, y:\mathrm{Y} \Rightarrow \underline{v}:\mathrm{Y}}\;\sigma \qquad \dfrac{\dfrac{\dfrac{}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \Rightarrow x:\mathrm{X}}\;\mathit{init}}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \Rightarrow \underline{u}:\mathrm{X}}\;\sigma \qquad \dfrac{\dfrac{\dfrac{}{x:\mathrm{X}, y:\mathrm{Y} \mid \mathrm{R}(x,y) \Rightarrow \mathrm{R}(x,y)}\;\mathit{init}}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \mid \mathrm{R}(x,y) \Rightarrow \mathrm{R}(x,\underline{v})}\;[\underline{v}\mapsto y]}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y}, \underline{u}:\mathrm{X} \mid \mathrm{R}(\underline{u},y) \Rightarrow \mathrm{R}(x,\underline{v})}\;[\underline{u}\mapsto x]}{x:\mathrm{X}, y:\mathrm{Y}, \underline{v}:\mathrm{Y} \mid \forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \mathrm{R}(x,\underline{v})}\;\forall\mathrm{L}}{x:\mathrm{X}, y:\mathrm{Y} \mid \forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}\;\exists\mathrm{R}}{x:\mathrm{X} \mid \exists y:\mathrm{Y}.\,\forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}\;\exists\mathrm{L}}{\emptyset \mid \exists y:\mathrm{Y}.\,\forall u:\mathrm{X}.\,\mathrm{R}(u,y) \Rightarrow \forall x:\mathrm{X}.\,\exists v:\mathrm{Y}.\,\mathrm{R}(x,v)}\;\forall\mathrm{R}
$$

Exercise: what goes wrong if we try to prove the converse sequent?

Equivalence to LJ

Indexed sequent calculus proves the same sequents as ordinary sequent calculus:

Theorem

Every indexed sequent proof *can be transformed into an ordinary sequent proof (and vice versa).*

But the indexed system has first-class **logic variables**, in the form of *obligation variables*.

This is very useful for proof search.

We have just pulled from our hat a fancier rabbit[2] than the one that went in.

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- Indexed Sequent Calculus
- Proof Search Strategies for Logic Programming

Application to Logic Programming

- A **proof search strategy** is a procedure for determining inferences to apply to the *frontier* of an incomplete *proof*.
- We can characterize the logic programming computation mechanisms of **SLD-resolution** and **uniform proof** as search strategies in the indexed sequent calculus.
- We can also adapt Andreoli's strategy of **focusing** [And92] to this system.
- These strategies form a sequence of increasing generality, with focusing a non-deterministically complete strategy for full first-order logic having a much reduced search space.

Semantic Analysis of Search Strategies

The adjunction-theoretic semantics for indexed sequent calculus gives us an *algebraic* and *uniform* understanding of properties of search strategies.

For example, we can justify the following features.

apply eigenrules eagerly: these are bijections of proof objects so their application cannot sacrifice provability.

purge redundant assumptions from contexts: the *principal formula* of left eigenrules need not be retained in the premises.

determine which rules are strictly commuting: these rules are essentially parallel: only one of the possible ordering need be tried.

These observations lead naturally to focusing strategies.

Conclusion

In summary:

- The categorical approach to proof theory gives us a better understanding of the abstract algebraic principles governing a logic.
- Proof-theoretic semantics for logic programming languages provides declarative descriptions of their computation mechanisms in terms of search strategies.
- The *composition of these two approaches* permits the algebraic analysis of logic programming languages.

Appendix

Natural Deduction

- Inference Rules
- Derivation Conversions

Sequent Calculus

- Structural Rules
- Logical Rules

Natural Deduction

- Inference Rules
- Derivation Conversions

Inference Rules

$$\frac{A \quad B}{A \wedge B}\ \wedge+ \qquad \frac{A \wedge B}{A}\ \wedge-_1 \qquad \frac{A \wedge B}{B}\ \wedge-_2$$

$$\frac{A}{A \vee B}\ \vee+_1 \qquad \frac{B}{A \vee B}\ \vee+_2 \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \hline \mathcal{D}_1 \\ \hline C\end{array} \quad \begin{array}{c}[B]\\ \hline \mathcal{D}_2 \\ \hline C\end{array}}{C}\ \vee-$$

$$\frac{}{\top}\ \top+$$

no rule for $\top-$

no rule for $\bot+$

$$\frac{\bot}{A}\ \bot-$$

Inference Rules ctd.

$$\dfrac{\dfrac{\dfrac{[\mathrm{A}]}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{A} \supset \mathrm{B}} \supset +$$

$$\dfrac{\mathrm{A} \supset \mathrm{B} \qquad \mathrm{A}}{\mathrm{B}} \supset -$$

$$\dfrac{\dfrac{\dfrac{[e : \mathrm{X}]}{\mathcal{D}}}{\mathrm{A}[x \mapsto e]}}{\forall x : \mathrm{X} \,.\, \mathrm{A}} \forall +^{\dagger}$$

$$\dfrac{\forall x : \mathrm{X} \,.\, \mathrm{A} \qquad t : \mathrm{X}}{\mathrm{A}[x \mapsto t]} \forall -$$

$$\dfrac{t : \mathrm{X} \qquad \mathrm{A}[x \mapsto t]}{\exists x : \mathrm{X} \,.\, \mathrm{A}} \exists +$$

$$\dfrac{\exists x : \mathrm{X} \,.\, \mathrm{A} \qquad \dfrac{\dfrac{[e : \mathrm{X}] \,,\, [\mathrm{A}[x \mapsto e]]}{\mathcal{D}}}{\mathrm{B}}}{\mathrm{B}} \exists -^{\dagger}$$

† e may not occur outside of $\mathcal{D}$ or in any open premise

- Inference Rules
- Derivation Conversions

Local Reductions

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{A_1} \quad \dfrac{\mathcal{D}_2}{A_2}}{A_1 \wedge A_2}\wedge+}{A_i}\wedge-_i \quad \overset{\wedge>}{\longmapsto} \quad \dfrac{\mathcal{D}_i}{A_i}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{E}_i}{A_i}}{A_1 \vee A_2}\vee+_i \quad \dfrac{\dfrac{[A_1]^u}{\mathcal{D}_1}}{C} \quad \dfrac{\dfrac{[A_2]^v}{\mathcal{D}_2}}{C}}{C}\vee-^{u,v} \quad \overset{\vee>}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}_i}{A_i}}{\mathcal{D}_i}}{C}$$

no local reduction for $\top$

no local reduction for $\bot$

Local Reductions ctd.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^u}{\mathcal{D}}}{B}}{A \supset B}\,{\supset}{+}^u \qquad \dfrac{\mathcal{E}}{A}}{B}\,{\supset}{-} \qquad \overset{\supset>}{\longmapsto} \qquad \dfrac{\dfrac{\dfrac{\mathcal{E}}{A}}{\mathcal{D}}}{B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[e : X]}{\mathcal{D}}}{A[x \mapsto e]}}{\forall x : X \,.\, A}\,\forall{+}^e \qquad \dfrac{\mathcal{T}}{t : X}}{A[x \mapsto t]}\,\forall{-} \qquad \overset{\forall>}{\longmapsto} \qquad \dfrac{\dfrac{\dfrac{\mathcal{T}}{t : X}}{\mathcal{D}[e \mapsto t]}}{A[x \mapsto t]}$$

$$\dfrac{\dfrac{\dfrac{\mathcal{T}}{t : X} \qquad \dfrac{\mathcal{E}}{A[x \mapsto t]}}{\exists x : X \,.\, A}\,\exists{+} \qquad \dfrac{\dfrac{[e : X] \qquad [A[x \mapsto e]]^u}{\mathcal{D}}}{B}}{B}\,\exists{-}^{e,u} \qquad \overset{\exists>}{\longmapsto} \qquad \dfrac{\dfrac{\dfrac{\mathcal{T}}{t : X} \qquad \dfrac{\mathcal{E}}{A[x \mapsto t]}}{\mathcal{D}[e \mapsto t]}}{B}$$

Local Expansions

$$\dfrac{\mathcal{E}}{A \wedge B} \quad \overset{\wedge<}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}}{A \wedge B}}{A}\wedge-_1 \qquad \dfrac{\dfrac{\mathcal{E}}{A \wedge B}}{B}\wedge-_2}{A \wedge B}\wedge+$$

$$\dfrac{\mathcal{E}}{A \vee B} \quad \overset{\vee<}{\longmapsto} \quad \dfrac{\dfrac{\mathcal{E}}{A \vee B} \qquad \dfrac{\overline{A}^{u}}{A \vee B}\vee+_1 \qquad \dfrac{\overline{B}^{v}}{A \vee B}\vee+_2}{A \vee B}\vee-^{u,v}$$

$$\dfrac{\mathcal{E}}{\top} \quad \overset{\top<}{\longmapsto} \quad \dfrac{}{\top}\top+$$

$$\dfrac{\mathcal{E}}{\bot} \quad \overset{\bot<}{\longmapsto} \quad \dfrac{\dfrac{\mathcal{E}}{\bot}}{\bot}\bot-$$

Local Expansions ctd.

$$\dfrac{\mathcal{E}}{A \supset B} \quad \overset{\supset<}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}}{A \supset B} \quad \dfrac{}{A}\,u}{B}\supset-}{A \supset B}\supset+^{u}$$

$$\dfrac{\mathcal{E}}{\forall x : X \,.\, A} \quad \overset{\forall<}{\longmapsto} \quad \dfrac{\dfrac{\dfrac{\mathcal{E}}{\forall x : X \,.\, A} \quad \dfrac{}{e : X}}{A[x \mapsto e]}\forall-}{\forall x : X \,.\, A}\forall+^{e}$$

$$\dfrac{\mathcal{E}}{\exists x : X \,.\, A} \quad \overset{\exists<}{\longmapsto} \quad \dfrac{\dfrac{\mathcal{E}}{\exists x : X \,.\, A} \quad \dfrac{\dfrac{}{e : X} \quad \dfrac{}{A[x \mapsto e]}\,u}{\exists x : X \,.\, A}\exists+}{\exists x : X \,.\, A}\exists-^{e,u}$$

Permutation Conversions

$$\cfrac{A \lor B \qquad \cfrac{\cfrac{[A]}{\mathcal{D}_1}}{C} \qquad \cfrac{\cfrac{[B]}{\mathcal{D}_2}}{C}}{\cfrac{\cfrac{C}{\mathcal{E}}}{D}}\;\lor- \quad \overset{\lor\rightleftarrows}{\longmapsto} \quad \cfrac{A \lor B \qquad \cfrac{\cfrac{\cfrac{\cfrac{[A]}{\mathcal{D}_1}}{C}}{\mathcal{E}}}{D} \qquad \cfrac{\cfrac{\cfrac{\cfrac{[B]}{\mathcal{D}_2}}{C}}{\mathcal{E}}}{D}}{D}\;\lor-$$

$$\cfrac{\cfrac{\cfrac{\bot}{A}\;\bot-}{\mathcal{E}}}{B} \quad \overset{\bot\rightleftarrows}{\longmapsto} \quad \cfrac{\bot}{B}\;\bot-$$

$$\cfrac{\exists x : X \,.\, A \qquad \cfrac{\cfrac{[e : X]\,,\,[A[x \mapsto e]]}{\mathcal{D}}}{B}}{\cfrac{\cfrac{B}{\mathcal{E}}}{C}}\;\exists- \quad \overset{\exists\rightleftarrows}{\longmapsto} \quad \cfrac{\exists x : X \,.\, A \qquad \cfrac{\cfrac{\cfrac{\cfrac{[e : X]\,,\,[A[x \mapsto e]]}{\mathcal{D}}}{B}}{\mathcal{E}}}{C}}{C}\;\exists-$$

Sequent Calculus

- Structural Rules
- Logical Rules

Structural Rules

$$\frac{\Gamma , A , A \Longrightarrow B}{\Gamma , A \Longrightarrow B}\ cL$$

$$\frac{\Gamma \Longrightarrow B}{\Gamma , A \Longrightarrow B}\ wL$$

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma , A \Longrightarrow B}{\Gamma \Longrightarrow B}\ cut$$

$$\frac{}{\Gamma , A \Longrightarrow A}\ init$$

- Structural Rules
- Logical Rules

Logical Rules

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B}\ \wedge R \qquad\qquad \frac{\Gamma, A, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C}\ \wedge L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B}\ \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \Longrightarrow C \qquad \Gamma, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C}\ \vee L$$

$$\frac{}{\Gamma \Longrightarrow \top}\ \top R$$

no rule for $\top L$

no rule for $\bot R$

$$\frac{}{\Gamma, \bot \Longrightarrow A}\ \bot L$$

Logical Rules ctd.

$$\frac{\Gamma , A \implies B}{\Gamma \implies A \supset B}\ \supset\!R \qquad \frac{\Gamma \implies A \qquad \Gamma , B \implies C}{\Gamma , A \supset B \implies C}\ \supset\!L$$

$$\frac{\Gamma \implies A[x \mapsto e]}{\Gamma \implies \forall x : X . A}\ \forall R^{\dagger} \qquad \frac{t : X \qquad \Gamma , A[x \mapsto t] \implies B}{\Gamma , \forall x : X . A \implies B}\ \forall L$$

$$\frac{t : X \qquad \Gamma \implies A[x \mapsto t]}{\Gamma \implies \exists x : X . A}\ \exists R \qquad \frac{\Gamma , A[x \mapsto e] \implies B}{\Gamma , \exists x : X . A \implies B}\ \exists L^{\dagger}$$

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