Leinster's globular theory of weak ω-categories

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These are my notes for a series of lectures that I gave at the Homotopy Type Theory seminar at Carnegie Mellon University in the spring of 2014. They are in outline form and lack much exposition. At some point, I may go back and flesh them out, but I'm releasing them now in this form in the hope that they will be helpful to the seminar participants.

The purpose of these lectures was to motivate and explain Tom Leinster's proposed definition of weak ω-category in a relatively straightforward way using categorical gadgetry of independent interest.

It was my hope that by the end of the lectures, audience members would be able to understand and appreciate the statement that,

> a weak ω-category is an algebra for the free operad-with-contraction in the category of globular sets.

For a more detailed development of the ideas presented here, please consult the sources cited in the references, and in particular, Leinster's excellent book [Lei03], upon which these notes are largely based.

1 Plain Multicategories

1.1 Basic Definitions

Definition (plain multicategory) A **plain multicategory** $\mathbb{C}$ has,

objects: a collection, $\mathbb{C}_0$

maps: for $A_1 , \cdots , A_n , B : \mathbb{C}$, a collection

$$\mathbb{C}(A_1 , \cdots , A_n \to B) \qquad\qquad \text{if } n = 0, \text{ we write } \quad \mathbb{C}(\emptyset \to B)$$

composition: for $A_i^j , B_i , C : \mathbb{C}$,

$$- \cdot - : (\mathbb{C}(A_1^1 , \cdots , A_1^{m_1} \to B_1) \times \cdots \times \mathbb{C}(A_n^1 , \cdots , A_n^{m_n} \to B_n)) \times \mathbb{C}(B_1 , \cdots , B_n \to C)$$

identities: for $A : \mathbb{C}$, a map,

$$\mathbb{C}(A \to A)$$

satisfying the laws of,

associativity:

$$(f_1^1 , \cdots , f_1^{m_1} , \cdots , f_n^1 , \cdots , f_n^{m_n}) \cdot (g_1 , \cdots , g_n \cdot h) = (((f_1^1 , \cdots , f_1^{m_1}) \cdot g_1) , \cdots , ((f_n^1 , \cdots , f_n^{m_n}) \cdot g_n)) \cdot h$$

unity: for $f : \mathbb{C}(A_1 , \cdots , A_n \to B)$,

$$(\mathrm{id}_{A_1} , \cdots , \mathrm{id}_{A_n}) \cdot f = f = f \cdot \mathrm{id}_B$$

1

Definition (plain operad)   A **plain operad** is a plain multicategory having just a single object:

$$\mathbb{C}_0 = \{\star\}$$

It is customary to abbreviate the hom sets of an operad by their arity:

$$\mathbb{A}(n) \quad := \quad \mathbb{A}\,(\underbrace{\star\,,\cdots,\,\star}_{n} \to \star)$$

Remark   Operads arose in algebra to represent collections of abstract untyped operations of various arities, closed under composition. In the algebraic setting, operads are often assumed to come with a symmetric action. In the categorical setting, we do not assume this (think of "plain" as a pun on "plane").

1.2   Graphical Representations

We can draw the morphisms of multicategories using their one-point suspensions.
Then $f : A_1 , \cdots, A_n \to B$ and $g : \emptyset \to C$ become:

and
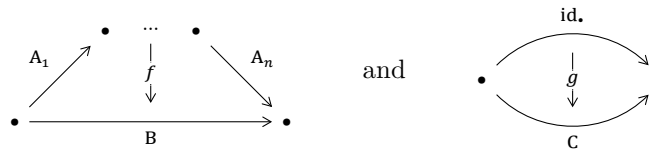
Or we can use the graph duals to draw them as string diagrams:

and
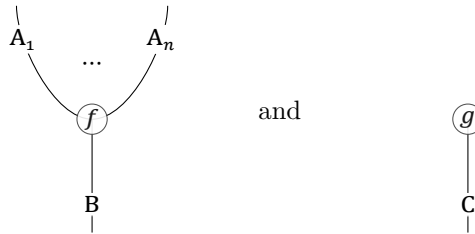
So compositions of morphisms in a multicategory is composition of labeled *rose trees*.

As string diagrams, there is no point in labeling the wires of an operad since they are all the same.

1.3   Multicategory Morphisms

Definition (multicategory morphism)   A **multicategory morphism**, $f : \mathbb{A} \to \mathbb{B}$ consists of a function on objects,

$$f_0 : \mathbb{A}_0 \to \mathbb{B}_0$$

together with a function on arrows,

$$f_1 : \mathbb{A}\,(A_1 , \cdots, A_n \to B) \to \mathbb{B}\,(f_0(A_1) , \cdots, f_0(A_n) \to f_0(B))$$

respecting composition and identity.

Definition (category of multicategories)   The category of (small) multicategories, MCAT, is defined in the obvious way.

Lemma (initial multicategory)   The empty multicategory, $\mathbb{0}$, having no objects, is initial.

Lemma (terminal multicategory)   The singleton multicategory, $\mathbb{1}$, having a single object, $\star$, and a single arrow in each hom set:

$$!_n : \mathbb{1}\,(\underbrace{\star,\cdots,\star}_{n} \to \star)$$

is terminal.

Definition (operad morphism)   An **operad morphism** is just a multicategory morphism between operads.

Definition (category of operads)   Thus the category of (small) operads, Opd, is a full subcategory of the category of (small) multicategories.

Lemma (initial operad)   The operad comprising only the identity operation is initial.

Lemma (terminal operad)   Since the terminal multicategory is an operad, it is also the terminal operad.

1.4   Multicategory Algebras

The concept of an operad came about to represent an *algebraic theory*, with abstract operations of various arities. The idea of an algebra for an operad is that of a *model* for such a theory.

Definition (operad algebra)   An **algebra** for an operad $\mathbb{C}$ is a set A together with a function θ sending the *abstract n-ary maps* of $\mathbb{C}$ to *concrete n-ary functions* on A:

$$f : \mathbb{C}(n) \qquad \mapsto \qquad \theta(f) : \underbrace{A \times \cdots \times A}_{n} \to A$$

in a way that respects composition and identity.

We can restate this definition in the language of multicategories and their morphisms by using the idea of the multicategory of sets:

Definition (multicategory of sets)   The **multicategory of sets**, Set$_\mathbf{M}$, has (small) sets for objects and hom sets,

$$\textsc{Set}_\mathbf{M}\,(A_1,\cdots,A_n \to B) \qquad \coloneqq \qquad \textsc{Set}\,(A_1 \times \cdots \times A_n \to B)$$

where "Set" is the cartesian *category* of sets and functions.

This way, we can say simply that an operad algebra is a *multicategory morphism* from the operad to Set$_\mathbf{M}$. This formulation generalizes to algebras for arbitrary multicategories:

Definition (multicategory algebra)   An **algebra** for a multicategory $\mathbb{C}$ is a multicategory morphism,

$$\theta : \textsc{MCat}\,(\mathbb{C} \to \textsc{Set}_\mathbf{M})$$

Example (sets)   The initial operad has as its algebras the sets.

$$\begin{aligned} 0 &\longrightarrow \textsc{Set}_\mathbf{M} \\ \star &\longmapsto X \\ \mathrm{id} : \star \longrightarrow \star &\longmapsto \mathrm{id} : X \to X \end{aligned}$$

Example (monoids)   The terminal operad (multicategory) has as its algebras the monoids.

$$\begin{aligned} 1 &\longrightarrow \textsc{Set}_\mathbf{M} \\ \star &\longmapsto X \\ m_n : \underbrace{\star,\cdots,\star}_{n} \longrightarrow \star &\longmapsto \mu_n : \underbrace{X \times \cdots \times X}_{n} \to X \end{aligned}$$

which, by associativity, are generates by the binary multiplication and nullary unit.

Example (semigroups)   The unique operad having a single operation of each *positive* arity has as its algebras the semigroups.

2   Generalized Multicategories

We can generalize the definition of multicategory to encompass that of other category-like structures, including ordinary categories, and relevant to our purposes, globular operads. In order to do this, we need to make a brief detour through internalization.

2.1   Internal Categories

Definition (bicategory of spans)   Let $\mathbb{C}$ be a category with pullbacks. The **bicategory of spans** on $\mathbb{C}$, $\textsc{Span}(\mathbb{C})$ has as

objects: those of $\mathbb{C}$.

1-cells: spans,

$$(d\,,c) : A \longrightarrow B \qquad \coloneqq \qquad \begin{array}{c} & S & \\ {}^{d}\swarrow & & \searrow^{c} \\ A & & B \end{array}$$

(mnemonic for "domain" and "codomain")

1-composition: spans formed by pullbacks of adjacent spans,

$$\begin{array}{c} & S & \\ {}^{d}\swarrow & & \searrow^{c} \\ A & & B \end{array} \cdot \begin{array}{c} & S' & \\ {}^{d'}\swarrow & & \searrow^{c'} \\ B & & C \end{array} \qquad \coloneqq \qquad \begin{array}{c} & & S\underset{B}{\times}S' & & \\ & \swarrow & \vee & \searrow & \\ & S & & S' & \\ {}^{d}\swarrow & \searrow^{c} & {}^{d'}\swarrow & & \searrow^{c'} \\ A & & B & & C \end{array}$$

1-identities: identity spans,

$$\text{id}_{A} : A \longrightarrow A \qquad \coloneqq \qquad \begin{array}{c} & A & \\ {}^{\text{id}}\swarrow & & \searrow^{\text{id}} \\ A & & A \end{array}$$

2-cells: maps of spans,

$$\alpha : (d\,,c) \longrightarrow (d'\,,c') \qquad \coloneqq \qquad \begin{array}{c} & S & \\ {}^{d}\swarrow & \big\downarrow{}^{\alpha} & \searrow^{c} \\ A & & B \\ {}^{d'}\nwarrow & \big\downarrow & \nearrow^{c'} \\ & S' & \end{array}$$

2-composition and identities: inherited from $\mathbb{C}$.

horizontal 2-composition: determined by the universal property of pullbacks,

$$\alpha \cdot\cdot \beta : (d_1 , c_1) \cdot (d_1' , c_1')) \longrightarrow (d_2 , c_2) \cdot (d_2' , c_2') \quad :=$$
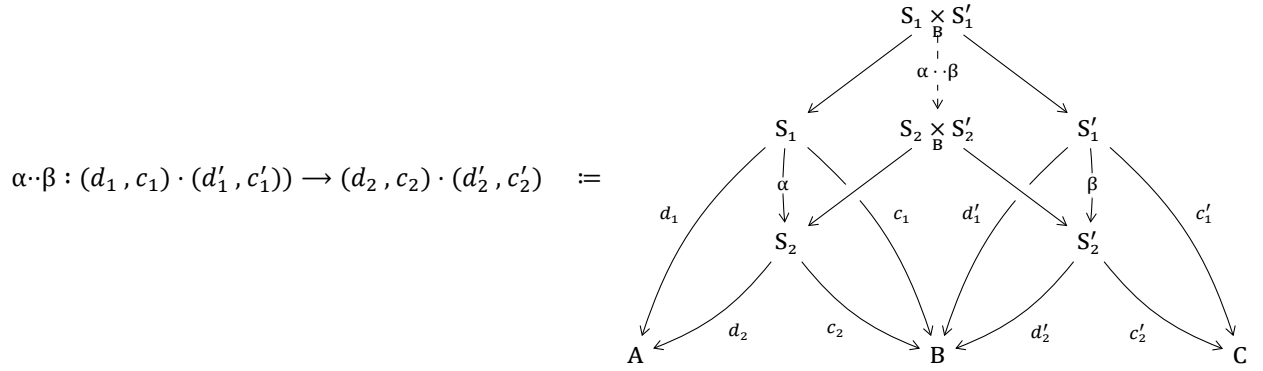
Remark (weakness) Because pullback is a universal property, 1-cell composition does not obey strict associative and unit laws, but rather obeys them only up to invertible 2-cells.

Definition (internal category) Let $\mathbb{C}$ be a category with pullbacks. An **internal category** in $\mathbb{C}$ is a monad in $\mathrm{SPAN}(\mathbb{C})$.
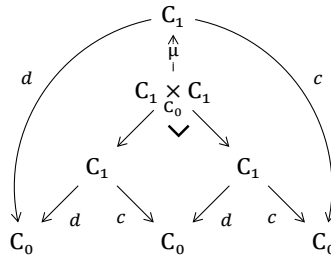
Explicitly, this consists of:

- A $\mathrm{SPAN}(\mathbb{C})$ object, C_0 (called the "object of objects")

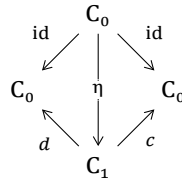- A $\mathrm{SPAN}(\mathbb{C})$ endomorphism, $(d , c) : C_0 \longrightarrow C_0$
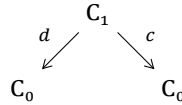
  (making C_1 an "object of arrows"):

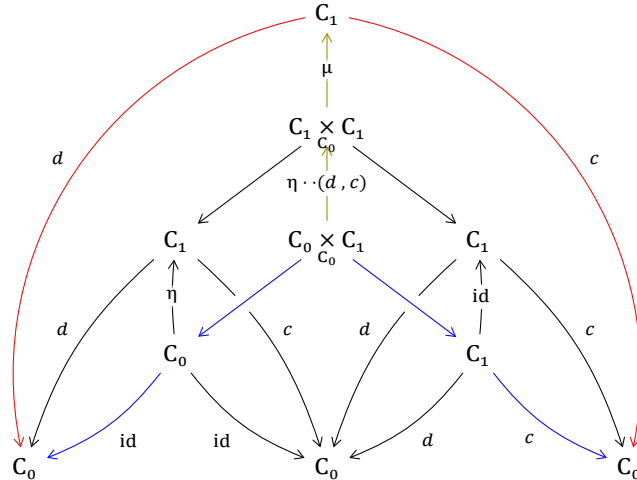- A $\mathrm{SPAN}(\mathbb{C})$ 2-cell, $\mu : (d , c) \cdot (d , c) \longrightarrow (d , c)$:

- A $\mathrm{SPAN}(\mathbb{C})$ 2-cell, $\eta : \mathrm{id}_{C_0} \longrightarrow (d , c)$:

- Obeying the associativity and unity **monad laws** in $\mathrm{SPAN}(\mathbb{C})$:

$$(\mu \cdot\cdot (d , c)) \cdot \mu = ((d , c) \cdot\cdot \mu) \cdot \mu \qquad\qquad (\eta \cdot\cdot (d , c)) \cdot \mu = \mathrm{id}_{(d,c)} = ((d , c) \cdot\cdot \eta) \cdot \mu$$

Lemma   The associativity and unity monad laws provide precisely the associativity and unity laws of composition in the internal category. Essentially, they strictify the weak structure of composition by providing a canonical *choice of pullbacks* (different choices would lead to isomorphic constructions). This can be seen by straightforward (but tedious) diagram chases. Here is the case for the (left) unit law, the associative law is left as exercises to the reader.
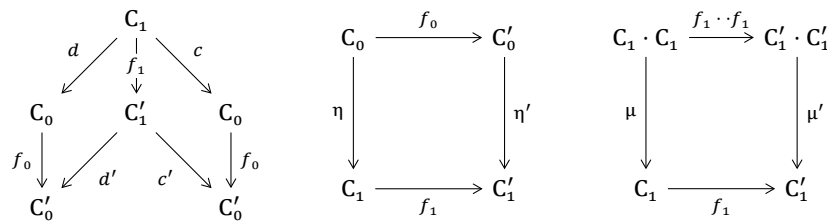
$$
\begin{array}{c}
C_1 \\
\xuparrow{\mu} \\
C_1 \underset{C_0}{\times} C_1 \\
\xrightarrow{\eta \cdot \cdot (d,c)} \\
C_1 \qquad C_0 \underset{C_0}{\times} C_1 \qquad C_1 \\
\eta \qquad\qquad\qquad \mathrm{id} \\
C_0 \qquad\qquad\qquad C_1 \\
C_0 \quad \mathrm{id} \quad \mathrm{id} \quad C_0 \quad d \quad c \quad C_0
\end{array}
$$

The monad left unit law along the spine (green) identifies precomposition by a unit (blue) with the identity (red).

Remark   In particular, an internal category in SET is a small category.

Definition (internal functor)   An internal category morphism, or **internal functor**, is a map between the respective spans that respects the composition structures.

Explicitly, $f : C \to C' \coloneqq (f_0 : C_0 \to C_0', \, f_1 : C_1 \to C_1')$ such that:

$$
\begin{array}{ccc}
\begin{array}{c}
C_1 \\
d \swarrow \;\; \downarrow f_1 \;\; \searrow c \\
C_0 \quad C_1' \quad C_0 \\
f_0 \downarrow \; d' \swarrow \;\; \searrow c' \; \downarrow f_0 \\
C_0' \qquad\qquad C_0'
\end{array}
&
\begin{array}{ccc}
C_0 & \xrightarrow{f_0} & C_0' \\
\eta \downarrow & & \downarrow \eta' \\
C_1 & \xrightarrow{f_1} & C_1'
\end{array}
&
\begin{array}{ccc}
C_1 \cdot C_1 & \xrightarrow{f_1 \cdot \cdot f_1} & C_1' \cdot C_1' \\
\mu \downarrow & & \downarrow \mu' \\
C_1 & \xrightarrow{f_1} & C_1'
\end{array}
\end{array}
$$

2.2   T-Multicategories

We can generalize this construction in a way that allows us to describe many category-like constructions, including plain multicategories and globular operads, by parameterizing by a cartesian monad.

Definition (cartesian monad)   A monad T is **cartesian** if:

- its natural transformations η and μ are **cartesian** (i.e. their naturality squares are pullbacks),
- its endofuctor T preserves pullbacks.

Remark   By the *two pullback lemma* a natural transformation is cartesian iff naturality squares with codomain 1 are pullbacks.

Lemma   The following monads are cartesian:

- The identity monad on any category

- The list monad (free monoid monad) on SET

- The "either monad" $(- + E)$ on SET (and hence the "maybe monad")

- The "state monad" $(- \times S)$ on SET

- The leaf-labeled rose-tree monad on SET

- The free monoidal category monad on CAT

Now we can generalize the construction of a bicategory of spans to that of a Kleisli bicategory of spans for a given cartesian monad. The only difference is that now the domain of each morphism is in the image of the monad's endofunctor.

Definition (Kleisli bicategory of spans) Let $\mathbb{C}$ be a category with pullbacks and T be a cartesian monad on $\mathbb{C}$. The **Kleisli bicategory of spans** of T on $\mathbb{C}$, $\text{SPAN}_T(\mathbb{C})$ has as

objects: those of $\mathbb{C}$.

1-cells: spans,

$$(d\,,c) : A \longrightarrow B \qquad := \qquad \begin{array}{c} S \\ {}^{d}\swarrow \quad \searrow^{c} \\ T(A) \qquad\quad B \end{array}$$

1-composition: spans formed by pullbacks of adjacent spans,

$$\begin{array}{c} S_1 \\ {}^{d_1}\swarrow\ \searrow^{c_1} \\ T(A)\qquad B \end{array} \quad\cdot\quad \begin{array}{c} S_2 \\ {}^{d_2}\swarrow\ \searrow^{c_2} \\ T(B)\qquad C \end{array} \qquad := \qquad \begin{array}{c} T(S_1)\underset{T(B)}{\times}S_2 \\ \vee \end{array}$$

1-identities: spans,

$$\text{id}_A : A \longrightarrow A \qquad := \qquad \begin{array}{c} A \\ {}^{\eta(A)}\swarrow\ \searrow^{\text{id}} \\ T(A)\qquad A \end{array}$$

2-cells: maps of spans.

2-composition and identities: inherited from $\mathbb{C}$.

horizontal 2-composition: determined by the universal property of pullbacks.

Remark (configurations) Intuitively, the endofunctor of the parametric monad, T, acts as a "type constructor for configurations", where a **configuration** is roughly a structured collection of things that we want to be able to combine together into a thing.

- Requiring a configuration to be a *functor* ensures that it is compatible with maps.

- Requiring that functor to support a *monad* lets us construct **singleton configurations** out of things (η), and **compound configurations** out of configurations of configurations of things (μ), which behave reasonably (monad laws).

- Requiring that the monad be *cartesian* is exactly what is needed to ensure that the coherence 2-cells (for associativity and units) are isomorphisms, and hence that T-spans do in fact form a bicategory.
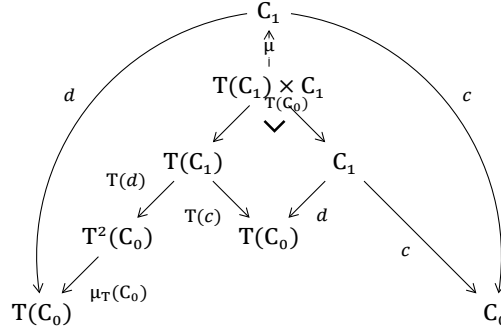
Now we can define generalized multicategories parameterized by a cartesian monad similarly to the way we defined internal categories before. Note that there are two distinct monads involved here: the **configuration monad**, T, a parametric cartesian monad *on* $\mathbb{C}$, and an internal monad *in* $\mathbb{C}$, determining the **composition structure** of the generalized multicategory.

Definition (generalized multicategory) Let $\mathbb{C}$ be a category with pullbacks and T be a *cartesian monad* on $\mathbb{C}$. An internal **T-multicategory** in $\mathbb{C}$ is a monad in the *Kleisli bicategory of spans*, $\textsc{Span}_T(\mathbb{C})$. Explicitly, this consists of:
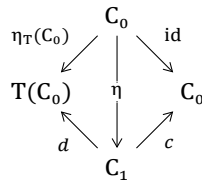
- A $\textsc{Span}_T(\mathbb{C})$ object, C_0

- A $\textsc{Span}_T(\mathbb{C})$ endomorphism, $(d, c) : C_0 \to C_0$:

$$
\begin{array}{ccc}
 & C_1 & \\
{}^{d}\swarrow & & \searrow{}^{c} \\
T(C_0) & & C_0
\end{array}
$$
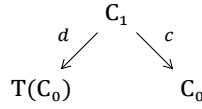
- A $\textsc{Span}_T(\mathbb{C})$ 2-cell, $\mu : (d, c) \cdot (d, c) \to (d, c)$:

- A $\textsc{Span}_T(\mathbb{C})$ 2-cell, $\eta : \mathrm{id}_{C_0} \to (d, c)$:

- Obeying the *monad laws* in $\textsc{Span}_T(\mathbb{C})$.

Remark Again, the associative and unit laws of the *internal monad* in $\textsc{Span}_T(\mathbb{C})$ determine the associative and unit laws of composition in the generalized multicategory.

Definition (generalized operad) T-multicategory on a terminal object of $\mathbb{C}$ is a **T-operad**.
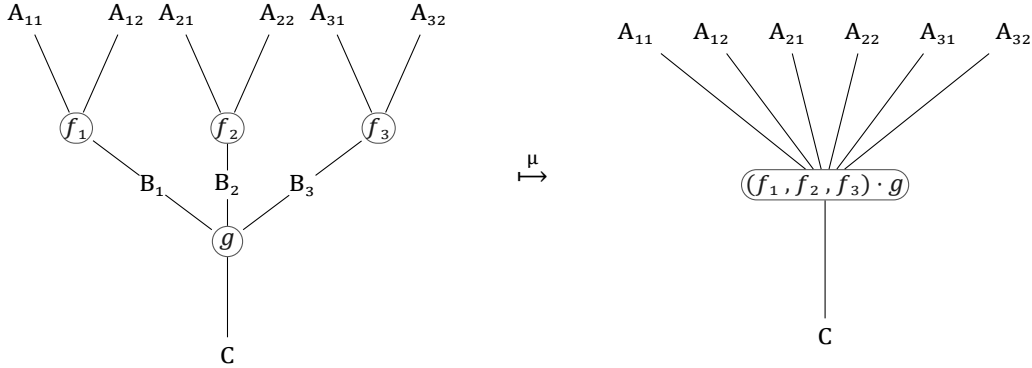
Remark In particular, in $\textsc{Set}$,

- an (id)-multicategory is a small category,

- an (id)-operad is a monoid,

- a $(-^*)$-multicategory is a *plain multicategory*,
- and a $(-^*)$-operad is a *plain operad*,

where $-^*$ is the free monoid (or list) monad.

Here is my attempt to represent the plain multicategory case:

The operation of the configuration monad, μ_T combines lists of lists of inputs into lists of inputs, while the operation of the internal monad μ combines a list of abstract operations f_1, f_2, f_3 and an abstract operation g into the abstract operation $(f_1, f_2, f_3) \cdot g$.
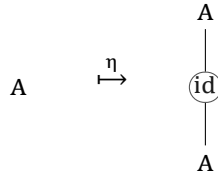
The unit of the configuration monad, η_T turns a thing into a singleton list of things to serve as the input to a unary operation, while the unit of the internal monad, η, takes a thing to the abstract operation that "unwraps" the singleton list containing it.

2.3 Generalized Multicategory Algebras

- The characterization of multicategory algebras as set-valued maps does not extend obviously to generalized multicategories.
- For an arbitrary cartesian monad on an arbitrary category with pullbacks there is not necessarily a sensible choice to act as the "generalized multicategory of sets".
- What is needed is an *internal* characterization of a generalized multicategory algebra, relying only on the ambient category with pullbacks, $\mathbb{C}$, and the cartesian configuration monad, T.

Recall that an *algebra for a plain operad* P is a set X together with a map $P \to n : \mathbb{N}. X^n \supset X$, or, by uncurrying, a map,

$$(\Sigma n : \mathbb{N}. P(n) \times X^n) \to X$$

Notice that this map has the structure of an *endofunctor algebra.*

So we define the endofuctor

$$
\begin{array}{ccc}
 & \dfrac{T_P}{\to} & \\
\text{SET} & \longrightarrow & \text{SET} \\
X & \longmapsto & \Sigma n : \mathbb{N}. P(n) \times X^n
\end{array}
$$

We can use the monad structure of the operad P to define a monad structure on T_P.

Suppressing arities, the multiplication, μ_P, has components:

$$T_P{}^2(X) = P(n) \times (P(k_1) \times X^{k_1}) \times \cdots \times (P(k_n) \times X^{k_n}) \cong P(n) \times P(k_1) \times \cdots \times P(k_n) \times X^{k_1 + \cdots + k_n}$$
$$\overset{\mu_P := \mu \times \mathrm{id}}{\longrightarrow}$$
$$P(m) \times X^m = T_P(X)$$

where μ is the composition induced by the internal monad determining the operad P.

The unit, η_P, has components:

$$\mathrm{id}_X \cong 1 \times X$$
$$\overset{\eta_P := \eta \times \mathrm{id}}{\longrightarrow}$$
$$P(1) \times X \hookrightarrow P(m) \times X^m = T_P(X)$$

Then we define the (plain) **operad algebras** of P to be the *monad algebras* of T_P:

$$\underbrace{\mathrm{ALG}(P)}_{\text{operad algebras}} \quad := \quad \underbrace{\mathrm{ALG}(T_P)}_{\text{monad algebras}}$$

We can do this for plain multicategories as well, and indeed for generalized multicategories (for the most general case, see [Lei03] §§6.2-4). The construction no longer refers to sets, but is completely internal. This provides a definition of **generalized multicategory algebra**.

The basic idea is to use the multiplication and unit of the configuration and internal monads of a generalized multicategory to induce a monad structure on $\mathbb{C}/C_0$. Then the algebras of this monad are taken as the algebras of the generalized multicategory. But the constructions are rather complex, so we don't present the details here (see also [Lei03] §4.3, [Lei00] §2.3 or [Lei97] §3).

As expected, we have the following special cases:

- The algebras for an (id)-multicategory on SET (i.e. for a plain category) are SET-valued functors.

- The algebras for a list-multicategory on SET (i.e. for a plain multicategory) are SET$_{\mathbf{M}}$-valued multicategory morphisms.

3 Globular Operads

3.1 Globular Sets

Like simplicial sets, cubical sets and opetopic sets, globular sets may be presented as presheaves on a **category of shapes**, in this case, globes:

Definition (globe category ($\bigcirc$))   The **globe category** has natural numbers for objects and arrows $\sigma_n, \tau_n : n \to succ(n)$, subject to the **globularity conditions**:

$$\sigma_n \cdot \sigma_{succ(n)} = \sigma_n \cdot \tau_{succ(n)} \qquad\qquad \tau_n \cdot \sigma_{succ(n)} = \tau_n \cdot \tau_{succ(n)}$$

Definition (globular set)   A **globular set** is a *presheaf* on the globe category:

$$A : \bigcirc^\circ \to \text{SET}$$

Explicitly, a globular set is given by a $\mathbb{N}$-indexed set, $\{A_n\}_{n \in \mathbb{N}}$, the sets of globular n-cells or n-**globes**, together with source and target **boundary maps**,

$$s_n, t_n : A_{succ(n)} \to A_n$$

subject to the *globularity conditions*:

$$s_{succ(n)} \cdot s_n = t_{succ(n)} \cdot s_n \qquad\qquad s_{succ(n)} \cdot t_n = t_{succ(n)} \cdot t_n$$

We will suppress the dimension subscript on the boundary maps when they can be inferred.

Remark (parallel cells)   The globularity conditions ensure that the boundary n-cells of a $(succ(n))$-cell have the same source and target as each other. We call such cells **parallel**. For example:

- All 0-cells are parallel by the power of *negative thinking* (using the one-point suspension):

$$A \qquad\qquad B \qquad\qquad A \Uparrow B$$

- Two 1-cells are parallel if they have the same source and target 0-cells:

$$f \Uparrow g$$

- Two 2-cells are parallel if they have the same source and target 1-cells:

$$\alpha \Uparrow \beta$$

Remark (iterated boundary)   Since it is only the last boundary map in a composition that matters, the globularity conditions allow us to speak unambiguously of the **m-fold source and target** of an n-cell as,

$$s^m := \underbrace{b \cdot ... \cdot b}_{m-1} \cdot s \qquad \text{and} \qquad t^m := \underbrace{b \cdot ... \cdot b}_{m-1} \cdot t \qquad \text{where} \qquad b \in \{s,t\} \qquad \text{for} \qquad 0 < m \le n$$

Equivalently, we may speak of the source and target of an n-cell in dimension m for $0 \le m < n$. The difference being *relative* vs. *absolute* dimension.

Definition (globular set morphism)   A **globular set morphism** $f : A \to B$ is a $\mathbb{N}$-indexed collection of maps, $f_n : A_n \to B_n$ that commute with the source and target maps:

$$\text{for } b \in \{s,t\}$$

Definition (category of globular sets)   The **category of globular sets**, GSet, is defined in the obvious way.

Lemma   In the category of globular sets,

initial globular set: the trivial globular set, which has no cells in any dimension is **initial**.

terminal globular set: the singleton globular set, which has just one cell in each dimension is **terminal**.

3.2 Strict ω-Categories

Strict ω-categories are built on top of an underlying globular set. Intuitively:

- there are n potential ways to compose two n-dimensional cells, namely, along any compatible shared lower-dimensional boundary cell

- each n-dimensional cell induces an identity cell in each higher dimension

Thus, a strict ω-category consists of an underlying globular set X together with operations for composition,

$$X_j \underset{X_i}{\times} X_j \longrightarrow X_j$$

and identity,

$$X_i \longrightarrow X_j$$

for all $i < j$, satisfying the appropriate boundary relations, and such that the compositions and identities obey strictly all the possible associative, unit and interchange laws. One way to make this precise is the following slick definition:

Definition (strict ω-category)   A **strict ω-category** consists of a *globular set*, X, having:

- for $i < j$, the structure of a *category* on $X_j \underset{t^{j-i}}{\overset{s^{j-i}}{\rightrightarrows}} X_i$,

- for $i < j < k$, the structure of a *strict 2-category* on $X_k \underset{t^{k-j}}{\overset{s^{k-j}}{\rightrightarrows}} X_j \underset{t^{j-i}}{\overset{s^{j-i}}{\rightrightarrows}} X_i$.

Remark

- The first condition ensures that j-cells composable in any allowed (i.e. lower) dimension have a *unique composite*, and that i-cells have a *unique identity* in any allowed (i.e. higher) dimension.

- The second condition ensures that all compositions satisfy the *interchange property*.

Together these conditions ensure that every n-dimensional pasting diagram has a unique composite n-cell.

Remark (notation)   We will index composition by *relative dimension* as follows:

$$
\begin{aligned}
f \cdot g &\ :\ A_i \underset{A_{i-1}}{\times} A_i \longrightarrow A_i \\
\alpha \cdot\cdot \beta &\ :\ A_i \underset{A_{i-2}}{\times} A_i \longrightarrow A_i \\
&\quad\ \vdots \\
\varphi \overset{(k)}{\cdot} \psi &\ :\ A_i \underset{A_{i-k}}{\times} A_i \longrightarrow A_i
\end{aligned}
$$

Definition (strict ω-category morphism)   A morphism of the underlying sets respecting the additional structure.

Definition (category of strict ω-categories)   The **category of strict ω-categories**, $\omega\mathrm{C{\small AT}}_{st}$, is defined in the obvious way.

Lemma

- There is an **underlying globular set functor** $U_\omega : \omega\mathrm{C{\small AT}}_{st} \longrightarrow \mathrm{G{\small SET}}$ that forgets the composition structure of a strict ω-category.

- This has a left adjoint, the **free strict ω-category functor** $F_\omega : \text{GSET} \longrightarrow \omega\text{CAT}_{st}$, which freely adds that structure to a globular set:

$$\text{GSET} \quad \perp \quad \omega\text{CAT}_{st}$$

with F_ω above and U_ω below.

- This adjunction is monadic and the induced monad *cartesian* (see [Lei03] appendix F).

Remark   This is like the adjunction between the underlying graph of a category and the free category on a graph, familiar from ordinary category theory.

Definition (free strict ω-category monad)   We call the induced monad on GSET, $T_\omega \coloneqq F_\omega \cdot U_\omega$, the **free strict ω-category monad**.

3.3   Globular Operads

Whereas a *plain operad* has a collection of abstract operations corresponding to each natural number *arity* a globular operad will have a collection of abstract operations corresponding to each globular *unlabeled pasting diagram*.

Definition (pasting diagrams)   The functor T_ω takes a globular set and creates formally all possible composites in it. Such formal composites are called **pasting diagrams**. Therefore, we will call the functor T_ω "pd" since it turns a globular set into the globular set of all pasting diagrams over it.

Remark (cell boundaries in pasting diagrams)   Since there is no *composition operation* in globular sets (only the formal composition provided by the pd functor), the source and target of each $(n+1)$-cell in a pasting diagram is each a *single n-cell*.

Example (pasting diagram)   Let X be a globular set having at least:

$$X(0) \supseteq \{A, B, C, D\}$$
$$X(1) \supseteq \{f_1, f_2, f_3, f_4, g, h_1, h_2\}$$
$$X(2) \supseteq \{\alpha_1, \alpha_2, \alpha_3, \gamma\}$$

and with boundary maps evident from the following. Here is a typical element of $pd(X)(2)$, that is, a 2-dimensional pasting diagram in X:
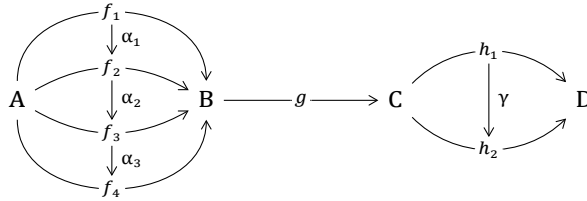

Definition (unlabeled pasting diagram)   Of particular interest is the globular set $pd(1)$, where 1 is the *terminal globular set*. The globular set $upd \coloneqq pd(1)$ is the globular set of **unlabeled pasting diagrams**.

Remark (boundaries of unlabeled pasting diagrams)   Since all n-cells of 1 are endomorphisms, the same must hold in upd. Thus instead of "source" and "target", we may speak simply of **boundary** (∂).
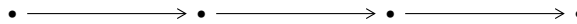
Example (unlabeled pasting diagram)   Here is a typical element of $upd(2)$, that is, a 2-dimensional unla-

beled pasting diagram:
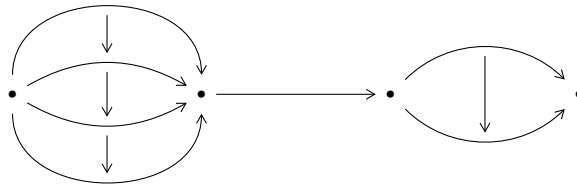
In fact it is the image under $pd(!) : pd(X) \to pd(1)$ of the pasting diagram of the previous example.

This has as boundary the 1-dimensional unlabeled pasting diagram:


Which in turn has as boundary the unlabeled 0-dimensional pasting diagram:

Remark (combinatorial presentations of unlabeled pasting diagrams)   Representing pasting diagrams in this way becomes challenging for higher dimensions. In the unlabeled case, there are equivalent combinatorial presentations, such as:

Batanin trees: The unlabeled 2-dimensional pasting diagram above can also be represented as an unlabeled tree by:
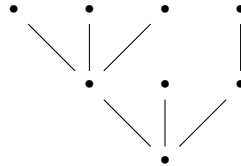
Leinster lists: or as a nested list over • by:

$$((\bullet, \bullet, \bullet), (), (\bullet))$$

The general principle may be inferred or found in [Lei03].

This last presentation makes plausible the following characterization of an unlabeled $(n+1)$-pasting diagram as a sequence of unlabeled n-pasting diagrams ([Lei03] §8.1).

Lemma

$$upd(0) = \bullet \qquad upd(n + 1) = (upd(n))^*$$

$$\partial : upd(n + 1) \to upd(n) = (\partial : upd(n) \to upd(n - 1))^*$$

The pd endofuctor on GSET lets us form formal compositions of cells, in the form of *pasting diagrams*. In fact, this functor supports the structure of a monad:

Definition (pasting diagrams monad)

- The multiplication of the monad takes a *compound pasting diagram* (i.e. pasting diagram of pasting diagrams) and "sews" it together into a pasting diagram. (cf: list concatenation for plain multicategories)
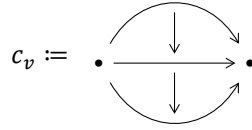
14

- The unit of the monad takes a cell and lifts it into a *singleton pasting diagram.* (cf: singleton list for plain multicategories)

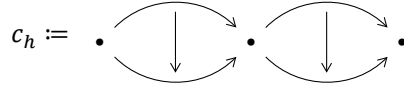This monad structure on pd determines a **composition structure on pasting diagrams**:

Example (composition of pasting diagrams)   Given diagrams π_1 , $\pi_2 : upd(2)$:

$$\pi_1 :=$$

$$\pi_2 :=$$
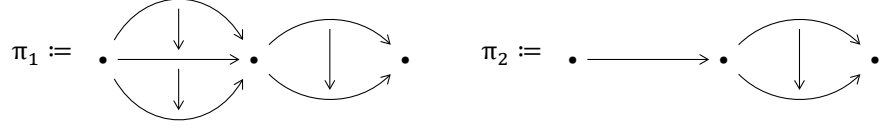
- their *vertical binary composition*, $\pi_1 \cdot \pi_2$, is specified by the diagram $c_v : pd(upd)(2)$:

$$c_v :=$$

- their *horizontal binary composition*, $\pi_1 \cdot\cdot \pi_2$, is specified by the diagram $c_h : pd(upd)(2)$:

$$c_h :=$$

- Furthermore, *identities* are subsumed by compositions as **nullary composition**, represented by the digram $c_n : pd(udp)(2)$:

$$c_n :=$$

We use pd as a *configuration monad* to construct *generalized operads* in the *category of globular sets* or "globular operads":

Definition (globular operad)   A **globular operad** is a pd-*multicategory* on the *terminal globular set* (which is thus necessarily an *operad*).

Explicitly, this consists of:

- A globular set, P, and *globular set morphism*, $d : \mathrm{P} \to pd(1)$, which determines an endomorphism on 1 in the *Kleisli bicategory of spans*,

$$(d, !) : \textsc{Span}_{pd}(\textsc{Gset})\,(1 \to 1) \qquad := \qquad \begin{array}{ccc} & \mathrm{P} & \\ {}^{d}\swarrow & & \searrow^{!} \\ pd(1) & & 1 \end{array}$$

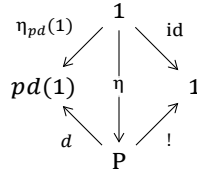- A $\textsc{Span}_{pd}(\textsc{GSet})$ 2-cell, $\mu : (d,!) \cdot (d,!) \to (d,!)$:

$$
\begin{array}{c}
P \\
\uparrow \mu \\
pd(P) \underset{pd(1)}{\times} P \\
\vee \\
pd(P) \qquad P \\
pd(d) \swarrow \quad pd(!) \searrow \quad \swarrow d \\
pd^2(1) \qquad pd(1) \\
\mu_{pd(1)} \\
pd(1) \qquad\qquad 1
\end{array}
$$

with outer arrows labelled d, $!$.

- A $\textsc{Span}_{pd}(\textsc{GSet})$ 2-cell, $\eta : \mathrm{id}_1 \to (d,!)$:

$$
\begin{array}{ccc}
 & 1 & \\
\eta_{pd(1)} \swarrow & \downarrow \eta & \searrow \mathrm{id} \\
pd(1) & & 1 \\
 d \nwarrow & \downarrow & \nearrow ! \\
 & P &
\end{array}
$$
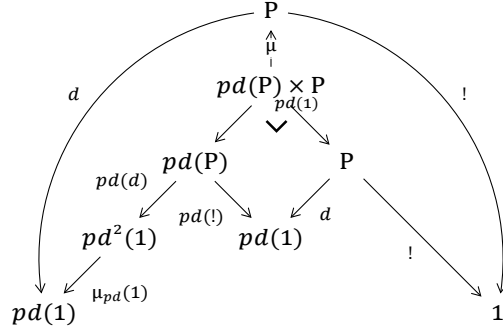
- Obeying the *monad laws* in $\textsc{Span}_{pd}(\textsc{GSet})$.

Remark

- The globular set P comprises the *abstract operations* of the globular operad.

- Abstract operations are indexed by unlabeled pasting diagrams or *globular configurations* by d.

- The fiber over an unlabeled pasting diagram is the set of abstract operations on that configuration.

- By convention, for unlabeled pasting diagram $\pi : upd$, the abstract operations on configuration π, $d^{-1}(\pi)$ is written "$P(\pi)$".

Here is an attempt to represent this graphically:
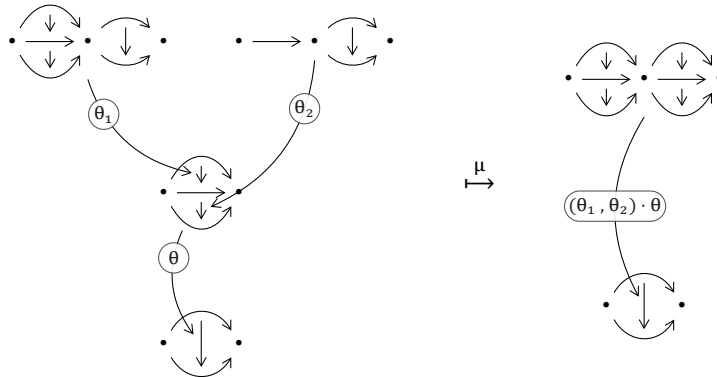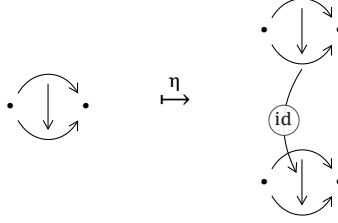
- (recall) The *multiplication of the configuration monad* of the operad combines a collection of pasting diagrams and a pasting diagram into a pasting diagram by *substitution*.

- The *internal multiplication* of the operad combines a collection of abstract operations and an abstract operation by *composition* in the operad.

- (recall) The *unit of the configuration monad* of the operad turns a cell into a singleton pasting diagram (containing just that cell).

- The *internal unit* of the operad turns a cell into the identity operation on (the singleton pasting diagram comprising) that cell.

- The operad associative law says that there is only one way of composing any 'tree' of abstract operations of the operad.

- The operad unit law says that the identity operation is unit for this combining.

In summary, a *globular operad* describes a collection of abstract operations, each combining an unlabeled pasting diagram into a cell, together with a *unique composite* for any family of operations that might plausibly be composed.

Definition (globular collection)   Since the "c" leg of an operad span is trivial, it is customarily suppressed. This motivates the notion of a **globular collection**, which is the globular set map, $d : \mathrm{P} \longrightarrow upd$, above.

Definition (category of globular collections)   Collections form a **category**, the slice category of globular sets over upd. The **morphisms** are commuting triangles.

Remark (underlying collection of an operad)   It will be useful in the sequel to distinguish the underlying collection of an operad from its composition structure, that is, μ, η, associativity and unity.

3.4   Globular Operad Algebras

if P is a globular operad then a P-algebra structure on a globular set X consists of a function sending pasting diagram labelings by cells of X to cells of X in a coherent way.

That is, it is a P-*action on* X, whereby the *abstract operations* of the operad P are mapped to *concrete operations* – i.e. GSET-morphisms sending X-pasting diagrams to X-cells.

Definition (globular operad algebra)   For globular operad P and globular set X, a **P-algebra on** X is a *monad algebra* for the monad pd_{P} on carrier X, where $pd_{\mathrm{P}}(\mathrm{X})$ is the pullback:

This is an instance of concept of *generalized multicategory algebra* mentioned before.

The functor $pd_\mathbf{P}$ combines *configurations* of cells of X with *abstract operations* from P in such a way that the domains of the abstract operations matches the configurations *labeled* by cells of X.

Thus a *monad algebra* for $pd_\mathbf{P}$ is an *endofunctor algebra* for $pd_\mathbf{P} : \textsc{GSet} \to \textsc{GSet}$, that is, a map $\mathit{struct} :$ $\textsc{GSet}\,(pd_\mathbf{P}(\mathsf{X}) \to \mathsf{X})$ behaving well with respect to composition.

This map struct then is an operation that is the *action* of the abstract operations of the operad P on appropriate configurations of cells in X, yielding cells in X.
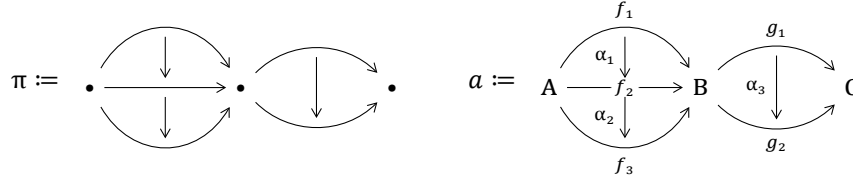
Morally, we can "curry" this map to:
$$\mathsf{P} \to pd(\mathsf{X}) \supset \mathsf{X}$$
so that the *abstract operations* of the operad map to *concrete operations* on X (in the form of morphisms $\textsc{GSet}\,(pd(\mathsf{X}) \to \mathsf{X})$).

We write "$\bar\theta$" for the concrete operation on X corresponding to the abstract operation θ in the operad P.

Example For $\pi : upd(2)$, $\theta : P(\pi)$ and $a : pd(\mathsf{X})(2)$ as follows:

Then $\bar\theta(a)$ is some 2-cell of X.

Example (strict ω-categories) The **terminal globular operad** has exactly one abstract operation for each configuration. An algebra for this operad is just a *strict ω-category*. So the terminal globular operad serves as the *theory of strict ω-categories*.

4 Weak ω-Categories

In a *strict ω-category*, there is precisely one way of composing a given pasting diagram. In contrast, in a *weak ω-category* there may be many ways, but the resulting cells are all equivalent in a suitably weak sense.

To summarize the situation we are in, we want to define a globular operad (thought of as a collection of abstract operations) to serve as the *theory* of weak ω-categories. Then weak ω-categories themselves will be *algebras* for this operad in the manner already explained. So the question that we need to answer is, *which abstract operations belong in this theory?*

Intuitively, the structure of a weak ω-category should be determined by two kinds of operations:

composition: For any pasting diagram of cells there should be an operation composing it to a cell in a manner consistent with any such composition of its boundary.

coherence: Any two ways of composing the same pasting diagram that agree on the composition of the boundary should give rise to a coherence cell relating the two composites.

The composition and coherence structure of any such theory is in turn dependent on the choice of **compositional bias**, which determines which composition operations (on pasting diagrams) are primitive, and which are derived by (operadic) composition (of composition operations).

Example (bias in $\textsc{Cat}$) In standard presentations of ordinary 1-dimensional categories, composition is biased toward $\{0,2\}$, meaning that we have a nullary composition operation (identity), and binary composition operation $(- \circ -)$, which are primitive, with all other compositions of arrows are built from these.

Alternatively, we could postulate an n-fold composition operation for each chain of arrows of length n. Such a presentation would be **unbiased**.

Due to the unit and associative laws, the two presentations are equivalent.

The most obvious (but a technically challenging) way to go about defining a theory of weak ω-categories is to set up a family of higher-dimensional biased *abstract composition operations* subject to a family of higher-dimensional *abstract coherence constraints*. This is the strategy followed by Batanin, but it is not Leinster's strategy. In Leinster's proposed construction, composition is unbiased, and furthermore, no distinction is made between composition and coherence. Rather, they emerge as two aspects of a single idea, called "contraction". Contraction alone is able to generate a natural theory of weak ω-categories.

4.1   Contractions

Leinster proposed that describing the *ways of composing* that are available in a weak ω-category should depend on one simple principle: the contraction principle. The contraction principle is a consequence of a *lifting property* that says basically that given a *globular collection* $d : P \longrightarrow upd$, and an unlabeled pasting diagram, every lift of its boundary to P by d must give rise to a specified lift of the diagram itself.

Definition (contraction principle)   Given an n-dimensional globular pasting diagram and ways of composing the $(n-1)$-dimensional diagrams at its source and target such that these coincide on the $(n-2)$-dimensional source and target (i.e. such that they are *parallel*), the **contraction principle** says that there is a *specified way of composing* the whole n-dimensional diagram inducing the given compositions on its boundary.

Example   Here are some applications of the contraction principle in low dimensions:

dimension 0 (trivial): For 0-dimensional singleton pasting diagram A, there is (by definition, or negative thinking) only one way to compose its -1-dimensional boundary, which is trivially parallel to itself. So the contraction principle says that there is a specified way of composing (the diagram) A to (the cell) A.

In general, the contraction principle gives a specified singleton composition – that is, a *specified identity operation* – for each 0-cell.

dimension 1 (composition-like): For 1-dimensional pasting diagram,

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \xrightarrow{\quad h \quad} D$$

there is only one way to compose the boundary 0-dimensional diagrams, namely as A and D, which are (trivially) parallel, so the contraction principle says that there is a specified way to compose the whole diagram to a 1-cell (call it $f \cdot g \cdot h : A \longrightarrow D$).
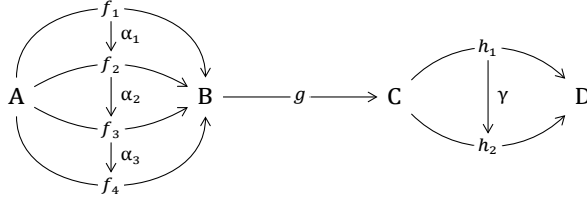
In general, the contraction principle gives a *specified composition operation* for any pasting diagram (n-ary chain) of 1-cells.

dimension 2 (coherence-like): We can view the above diagram as a trivial 2-dimensional pasting diagram. Contraction (providing binary compositions) together with operad structure (providing subdiagram substitutions) gives us both the 1-cells $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$. These two 1-cells are parallel, having boundary $A \longrightarrow D$, so contraction also gives us a specified 2-cell witnessing the associativity of binary composition.
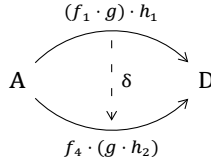
$$
\begin{array}{c}
(f \cdot g) \cdot h \\
A \quad \overset{\alpha}{\Longrightarrow} \quad D \\
f \cdot (g \cdot h)
\end{array}
$$

In general, the contraction principle gives a *specified coherence* 2-*cell* relating any two ways of composing a given diagram of 1-cells.

**dimension 2 (composition-like):** For 2-dimensional pasting diagram,

the 1-dimensional source composition, $(f_1 \cdot g) \cdot h_1$, and target composition, $f_4 \cdot (g \cdot h_2)$, agree on *their* boundary, $A \longrightarrow D$, so the contraction principle says there is a specified way to compose the whole diagram to a cell with the given boundary compositions:
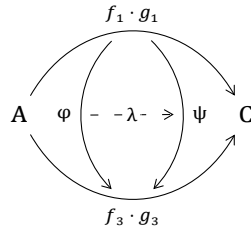

In general, the contraction principle gives a specified composition to each 2-dimensional pasting diagram for any chosen compositions of its boundary.

dimension 3 (coherence-like): We can view the 2-dimensional pasting diagram,
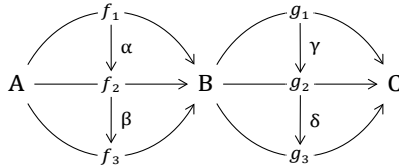
as a trivial 3-*dimensional* pasting diagram. One way to compose its 2-dimensional "source" is $\varphi \coloneqq (\alpha \cdot \cdot \gamma) \cdot (\beta \cdot \cdot \delta)$. Another way to compose its 2-dimensional "target" is $\psi \coloneqq (\alpha \cdot \beta) \cdot \cdot (\gamma \cdot \delta)$. These agree on their 1-dimensional boundary, $f_1 \cdot g_1 \longrightarrow f_3 \cdot g_3$, so the contraction principle says that there is a specified 3-cell in $\varphi \longrightarrow \psi$:

corresponding to the coherence of **interchange** in a bicategory.

In general, given any 2-dimensional pasting diagram and fixed way of composing its 1-dimensional boundary, the contraction principle gives a 3-dimensional coherence cell relating any chosen ways of composing the diagram (consistent with the chosen way of composing its boundary).

The contraction principle subsumes the two (apparently distinct) operations of *composition* and *coherence*. Leinster proposes that the abstract operations of a weak ω-category be generated by using (only) the contraction principle and the structure of an operad.

Remark (free and unbiased)   This proposal is as *free* and *unbiased* as possible, in the sense that it doesn't pick out some distinguished subset of compositions and coherences as "primitive" and require that the rest be composed out of these. For example we do *not* require that:

$$\alpha : (f \cdot g) \cdot h \longrightarrow f \cdot (g \cdot h) := (\alpha_l : (f \cdot g) \cdot h \longrightarrow f \cdot g \cdot h) \cdot (\alpha_r{}^{-1} : f \cdot g \cdot h \longrightarrow f \cdot (g \cdot h))$$

Rather, each possible pair of ways to compose the diagram results in a distinct coherence. However, because they are parallel, α and $\alpha_l \cdot \alpha_r{}^{-1}$ will themselves be related by a coherence one level up.

- A *way of composing* a pasting diagram given a way of composing its boundary is an *abstract operation* on its configuration.

- The structure encoding all of these ways of composing pasting diagrams is a *globular operad*. The composition structure of the operad determines derived abstract operations, for example, that first compose a subdiagram and then compose the whole diagram with the subdiagram substituted by its composite.

- A *coherence operation* determines a cell bounded by two ways of composing the same diagram.

- In a *weak* ω-category there are generally multiple ways to compose a diagram, but each pair of such is related by a coherence.

The *contraction principle* can be justified by a structure called a contraction on a globular operad, which is an instance of a contraction on a morphism of globular sets:

Definition (contraction on a globular set morphism)   A **contraction on a globular set morphism** $p : X \longrightarrow Y$ is a *lifting* by p of Y cells to X cells by their boundaries.

Explicitly, it is a function κ that assigns to $\varphi : Y(n+1)$ and *parallel* $f \Uparrow g : X(n)$ with $p(f) = s(\varphi)$ and $p(g) = t(\varphi)$ a cell in $X(n+1)$ such that $p(\kappa_\varphi(f, g)) = \varphi$ (and thus $s(\kappa_\varphi(f, g)) = f$ and $t(\kappa_\varphi(f, g)) = g$).
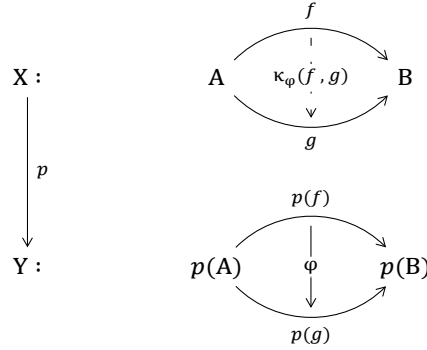
Definition (contraction morphism)   A **contraction morphism** is a morphism in the arrow category of globular sets that preserves the contraction structure.

Explicitly, given contractions κ on $p : X \longrightarrow Y$ and κ' on $p' : X' \longrightarrow Y'$, a map $\lambda : \kappa \longrightarrow \kappa'$ is a pair of globular set morphisms $\alpha : X \longrightarrow X', \beta : Y \longrightarrow Y'$ such that for any cells $f, g : X, \varphi : Y$ as above:

$$\kappa'_{\beta(\varphi)}(\alpha(f), \alpha(g)) = \alpha(\kappa_\varphi(f, g))$$

Definition (contraction on an operad)   A **contraction on an operad** is a *contraction* on its underlying *collection* $d : X \longrightarrow pd(1)$.

Explicitly, this is a map that sends an unlabeled pasting diagram and abstract operations labeling its boundary to an abstract operation labeling the diagram itself.
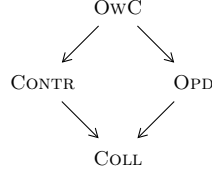
Remark (contraction on a collection)   Note that the contraction structure on a globular operad is really just a contraction on its underlying *collection*; that is, it does not interact with the composition structure of the operad (μ, η, associativity, unity).

Definition (operad-with-contraction)   An **operad-with-contraction** is a *globular collection* equipped with both the structure of an *operad* and that of a *contraction*.

Definition (operad-with-contraction morphism)   An **operad-with-contraction morphism** is both a *operad morphism* and a *contraction morphism* on its underlying *globular collection*

Definition (category of operads-with-contraction)   The **category of operads-with-contraction**, OwC, is defined in the obvious way.

We have forgetful functors (in Cat):

$$
\begin{array}{ccc}
 & \text{OwC} & \\
\swarrow & & \searrow \\
\text{Contr} & & \text{Opd} \\
\searrow & & \swarrow \\
 & \text{Coll} &
\end{array}
$$

This gives us an underlying functor $G : \text{OwC} \to \text{Coll}$. Since Coll is *locally finitely presentable*, G has a left adjoint F, which necessarily preserves colimits. Applying F to the initial (i.e. empty) collection, $¡ : \emptyset \to upd$ gives us an initial object, $(L, \lambda) : \text{OwC}$, where L is the operad and λ the specified contraction.

Definition (free operad with contraction)   The operad L with contraction λ is the **freely generated operad with contraction**. It acts as the **theory of weak ω-categories**.

Remark (inductive presentation of free operad with contraction)   Leinster establishes the existence of (L, λ) by universal properties [Lei03] appendix G, but notes that an inductive construction should be possible. In general, if L is known up to and including dimension $n - 1$, then $L(n)$ is obtained by first closing under contraction then closing under n-dimensional operadic composition and identities.

Cheng [Che08] describes a dimension-wise inductive presentation of the free operad-with-contraction functor, F out of the free operad and free contraction functors, each of which supports a monad. She then applies this to the initial globular collection to obtain L. This works because the structure in each dimension is stable with respect to the structure at higher dimensions and in the definition of operad-with-contraction there are no axioms governing the interaction of these two types of structure; therefore, the monad for operads and the monad for contractions may be combined without the need for a distributive law between them. Thus, the square is in fact a pullback (because there are no interactions between the contraction structure and the operad structure).

4.2   Operad-with-Contraction Algebras

Definition (weak ω-category)   A **weak ω-category** is an algebra for L.

Definition (category of weak ω-categories)   The **category of weak ω-categories** (ωCat_{wk}) is just the category of algebras for L.

Remark (weak ω-category morphisms)   Counterintuitively, maps in ωCat_{wk} (being operad algebra morphisms) preserve the operations from L – that is, the weak ω-category structure – strictly. A definition of *weak ω-functor* in this context is still an open problem (as far as I know).

Remark (strict ω-categories)   The *terminal operad* admits a unique contraction. This yields the **terminal operad-with-contraction**, and its algebras are the *strict ω-categories*.

Furthermore, the algebra construction is functorial and contravariant, so the unique map $L \to 1$ induces a canonical functor,

$$\text{Alg}(1) = \omega\text{Cat}_{st} \to \omega\text{Cat}_{wk} = \text{Alg}(L)$$

which is full and faithful.

Thus, every strict ω-category is a weak ω-category, as we would expect.

References

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